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A theory of computation based on unsharp quantum logic: Finite state automata and pushdown automata[☆]

Yun Shang[a,*], Xian Lu[a], Ruqian Lu[a,b]

[a] *Institute of Mathematics, AMSS, CAS, Beijing 100190, PR China*
[b] *CAS Key Lab of IIP, Institute of Computing Technology, CAS, PR China*

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A B S T R A C T

When generalizing the projection-valued measurements to the positive operator-valued measurements, the notion of the quantum logic generalizes from the sharp quantum logic to the unsharp quantum logic. It is known that: (i) the distributive law is one of the main differences between the sharp quantum logic and the boolean logic, and the block or the center of the sharp quantum structures are boolean algebras; (ii) the unsharp quantum logic does not satisfy the non-contradiction law, which forces the block or the center of unsharp quantum structures to be multiple valued algebras, rather than boolean algebras. Multiple valued algebras, as special quantum structures, are the algebraic semantics of multiple valued logic. Interestingly, we recently discovered that the difference between some unsharp quantum structures and multiple valued algebras is also some kind of distributive law.

Choosing an orthomodular lattice (an algebraic model of a sharp quantum logic) to be the truth valued lattice, Ying et al. have systematically developed automata theory based on sharp quantum logic. In this paper, choosing a lattice ordered quantum multiple valued algebra $\mathcal{E}$ (an extended lattice ordered effect algebra $\mathcal{E}$, respectively) to be the truth valued lattice, we also systematically develop an automata theory based on unsharp quantum logic. We introduce $\mathcal{E}$-valued finite-state automata and $\mathcal{E}$-valued pushdown automata in the framework of unsharp quantum logic. We study the classes of languages accepted by these automata and re-examine their various properties in the framework of unsharp quantum logic. The study includes the equivalence between finite-state automata and regular expressions, as well as the equivalence between pushdown automata and context-free grammars. It is also demonstrated that the universal validity of some important properties (such as some closure properties of languages and Kleene theorem etc.) depends heavily on the aforementioned distributive law. More precisely, when the underlying model degenerates into an MV algebra, then all the counterparts of properties in classical automata are valid. This is the main difference between automata theory based on unsharp quantum logic and automata theory based on sharp quantum logic.

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1. Introduction

Quantum logic was introduced by Birkhoff and von Neumann [1] in 1930s as the logic of quantum mechanics, where the projection operators (onto closed subspaces) of a Hilbert space are regarded as quantum events of the logic. In closed

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* Corresponding author. Tel.: +86 01062651452.
E-mail addresses: shangyun602@163.com (Y. Shang), luxian@amss.ac.cn (X. Lu), rqlu@math.ac.cn (R. Lu).

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quantum systems, where the quantum processes are reversible, quantum events correspond to the projection-valued (PV) measurement of an observable. Since the set $\mathcal{P}(\mathcal{H})$ of all projection operators of a separable infinite dimensional Hilbert space is an orthomodular lattice [19], orthomodular lattices have been the main algebraic model in the study of quantum logic. However, the set of the projection operators is not the set of maximal possible events, according to the Born statistical rule of quantum theory, which may happen in the theory of irreversible quantum processes. To generalize the quantum theory accordingly, the PV measurements are generalized to the positive operator valued (POV) measurements [2]. We write $\mathcal{E}(\mathcal{H})$ for the set of all positive operators dominated by identity on Hilbert space, and its elements are called effects. Since any event in $\mathcal{P}(\mathcal{H})$ always satisfies the non-contradiction principle, such an event is called sharp. Quantum logic corresponding to $\mathcal{P}(\mathcal{H})$ is called sharp quantum logic. Quantum events depicted by $\mathcal{E}(\mathcal{H})$ do not satisfy non-contradiction principle. These events are called unsharp events, and the quantum logic corresponding to $\mathcal{E}(\mathcal{H})$ is called unsharp quantum logic [6].

In the recent years, many algebraic structures were proposed to characterize quantum effects. In 1994, Foulis introduced effect algebras, which is the main model for unsharp quantum logic. At the same time, from the point of view of fuzzy sets, Kôpka and Chovanec [23] introduced *D*-posets, which are equivalent to effect algebras. They are both equivalent to weak orthoalgebras [11]. In the study of the unsharp quantum logic, MV algebras, as the algebraic model of multiple valued logic, play an analogous role to that of boolean algebras in sharp quantum logic [5,8]. They are the blocks of lattice ordered effect algebras (or lattice ordered difference posets) [35]. Quantum MV (QMV) algebras form another important type of unsharp quantum structures [12]. They are not only non-lattice theoretic generalizations of MV algebras, but also non-idempotent generalizations of orthomodular lattices. It is known that an MV algebra is the center of a QMV algebra [10,13].

In the classical computing theory, finite-state automata and pushdown automata are the basic mathematical models of computing which play an important role in designing and implementing programming languages [17]. Their logical foundation is the boolean logic. A number of different quantum automata models, such as finite quantum automata and pushdown quantum automata, were proposed as mathematical models of quantum computing. Some models of quantum automata were proposed from the probabilistic point of view [30], where each transition is assigned a function whose value is interpreted as the probability amplitude of the transition [22,29,15]. These quantum automata may be viewed as computing models based on quantum mechanics. Logical foundation of these models, as well as of the quantum Turing machine, is still the boolean logic. Since quantum logic is different from the classical boolean logic, a quantum device should obey its own logic [41].

In order to develop a computing theory based on quantum logic, some models of quantum automata were proposed from the quantum logical point of view [38–41,31–34]. The set of truth values of the logic is understood as an orthomodular lattice and an element of the orthomodular lattice is assigned to each transition of an automaton. Namely, it is considered as a truth value of the proposition describing the transition. With this approach, the authors revealed an essential difference between classical computation theory based on boolean logic and computation theory based on sharp quantum logic. They found that many important properties of automata depend heavily on the distributivity of underlying logic. That is, some important properties are universally valid if and only if the underlying logic is a boolean algebra [40,41].

From the relation between sharp quantum logic and unsharp quantum logic, we can obviously learn that computation theory based on unsharp quantum logic is more general. Recently, the theory of open quantum system has drawn the attention of many physicists. It has been applied extensively in various fields of quantum physics such as quantum stochastic processes, quantum optics, (non)relativistic quantum theory, and quantum information science [7,20]. The theory of POV measurement has been developed and successful applications in the aforementioned topics [4] have been found. Because of these reasons, we hope that our computation theory based on unsharp quantum logic may provide some logical foundation for open quantum systems.

This paper is a systematic exposition of automata theory based on unsharp quantum logic. We mainly consider two algebraic models of unsharp quantum logic — the extended lattice ordered effect algebras and the lattice ordered QMV algebras. We call them $\mathcal{E}$-valued lattice, where $\mathcal{E}$ stands for some extended effect algebra. An extended lattice ordered effect algebras is equivalent to a quasilinear QMV algebra [8,37]. In a related work, we have found that the main difference between them and MV algebras is some kind of distributive law [37]. This paper is the continuation of [37], and the new results include the following: (1) We introduce a new language recognition mode for unsharp quantum automata. This mode is called the parallel recognition as opposed to the sequential recognition discussed in [37,40,41]. We call the former as width-first recognition and the latter depth-first. The width-first recognition of a regular quantum language introduced in [26] was only for finite quantum automata with the sharp quantum logic, and in this paper we generalize it to unsharp quantum logic, as well as to other quantum automata such as pushdown quantum automata. We prove in this paper that the parallel recognizability of a quantum language is always equivalent or weaker than its sequential recognizability by the same unsharp quantum automaton. It is also demonstrated that the equivalence of these two recognizability powers depends heavily on the validity of the distributive law: The equality holds if and only if the underlying QMV algebra degenerates into an MV algebra. This is a characteristic difference between unsharp quantum automata and classical automata. For the case of sharp quantum automata, this difference was demonstrated in [26], but at a different level (orthomodular lattice). Furthermore, we have applied this result again to the sharp quantum automata and improved Proposition 2.6 in [26] by showing that the sequential recognizability of a sharp automaton is equal to its parallel recognizability if and only if the underlying orthomodular lattice degenerates into a boolean algebra.

(2) Basing on the width-first recognition type, we explore the language properties of $\mathcal{E}$-valued nondeterministic finite quantum automata ($\mathcal{E}$NFA) once again. They were studied in [37] for depth-first recognition only and consist of the closure properties of quantum languages such as intersection, disjoint sum, concatenation and reversal operation. Our research results show that some of the properties, e.g. those related to the intersection, union, concatenation and the reversal operation of the sharp quantum languages remain valid in the unsharp case as well. But there are also some important properties, which are valid for sharp quantum languages but no longer true for the unsharp case, such as that related to the complementation.

(3) The algebraic structure of sharp quantum logic is the orthomodular lattice, which allows the complementation to satisfy the non-contradiction law [8]. Equivalently, each element of the orthomodular lattice should be an idempotent under the disjoint sum operation, when the orthomodular lattice is changed into a QMV algebra. However, there may be elements of a QMV algebra, which are not idempotent under the disjoint sum operation. As a result, we show in this paper that the quantum languages based on a QMV algebra do not enjoy the closure property with respect to complementation. This is a characteristic difference between unsharp quantum automata and sharp quantum automata.

(4) We have proved in [37] that (by depth-first recognition) there are $\mathcal{E}$NFA for which there do not exist any $\mathcal{E}$DFA simulating them exactly, unless the underlying QMV algebra degenerates into an MV algebra. In this paper we prove that, under additional conditions, for each $\mathcal{E}$NFA there is a $\mathcal{E}$DFA simulating it exactly, provided that the principle of width-first recognition is applied.

(5) Similarly, though we do not know whether (by depth-first recognition) the traditional pumping lemma holds for unsharp quantum automata in general, we do prove in this paper that, with additional condition, the pumping lemma does hold for unsharp quantum automata provided that the principle of width-first recognition is applied.

(6) ϵ-moves and ϵ-reductions have been extensively studied for classical automata, and also for sharp quantum automata [40,41]. This paper, for the first time, studies the properties of ϵ-moves and ϵ-reduction for $\mathcal{E}$-valued nondeterministic finite state quantum automata. It is proved in this paper that, with additional conditions, each $\mathcal{E}$-valued nondeterministic finite-state automaton with ϵ-moves can be simulated by an $\mathcal{E}$-valued nondeterministic finite-state automaton exactly, provided that the principle of width-first recognition is applied.

(7) The Kleene theorem is one of the well-known results in the classical automata theory and has been extensively discussed for sharp quantum automata [40,41]. In this paper, we re-examined the Kleene theorem for unsharp quantum automata. It is proved that $\mathcal{E}$NFA and their $\mathcal{E}$-valued regular expressions have the same power of accepting languages if and only if the underlying $\mathcal{E}$-valued lattice degenerates into an MV algebra.

(8) The sharp version of the quantum pushdown automata was discussed in [40,41]. In this paper we developed a theory for $\mathcal{E}$-valued pushdown automata ($\mathcal{E}$PDA) together with their final state acceptance variant and empty stack acceptance variant. We show that these two variants are equivalent under the depth-first recognition mode. In the case of width-first recognition, we have the same result only when the underlying QMV algebra degenerates into an MV algebra. Analogously, we systematically study $\mathcal{E}$-valued context free grammars ($\mathcal{E}$CFG) and their Greibach and Chomsky normal forms. We show that these two normal forms are equivalent under the depth-first recognition mode. In case of the width-first recognition, as we might expect, the same result holds only if the underlying QMV algebra degenerates into an MV algebra. We also show the equivalence between $\mathcal{E}$PDA and $\mathcal{E}$CFG under depth-first recognition mode. In case of width-first recognition, we have the same result only when the underlying QMV algebra degenerates into an MV algebra.

By re-examining various properties of automata including the equivalence between the finite-state automata and regular expression and equivalence between the pushdown automata and context-free language, it is found that the general validity of many properties depend heavily on the above distributivity of $\mathcal{E}$-valued lattices. That is, when the $\mathcal{E}$-valued lattice becomes an MV algebra, almost all the counterparts of properties in classical automata are valid. This is the essential difference between automata theory based on unsharp quantum logic and classical automata theory.

A fundamental difference between quantum and classical mechanics is the fact that not all pairs of observables are jointly measurable. The notion of POV measurement offers a possibility for many families of observables to be jointly measurable and it becomes important to determine what price is to be paid for reconciling this classical feature with the underlying quantum structure [3,24]. In a wider context, the concept of coexistence of observables is used to describe the physical possibility of measuring together two or more quantities simultaneously [16,27]. On the other hand, we know that each MV algebra could be viewed as a lattice ordered effect algebra so that all elements are mutually compatible [8,35]. Physically, the compatible elements are called coexistent [16,27].

Our results imply that, when all elements of the underlying QMV algebra coexist, almost all the counterparts of properties in classical automata are valid. Since each effect determines a simple observable [3], it further hints that, in order to preserve the classical properties of automata in the physical implementation of quantum automata, it is sufficient and necessary that the simple observables corresponding to effects are coexistent. In the case of the sharp quantum logic, the underlying logic then degenerates into a boolean algebra. It requires that the simple observables corresponding to projection mutually commute, namely, they are simultaneously measurable [40,41]. Obviously, the restriction placed on observables in unsharp quantum logic is much weaker than that in sharp quantum logic.

The paper is organized as follows: In Section 2, we introduce some algebraic results used in this paper and give the definitions of $\mathcal{E}$-valued nondeterministic automata, $\mathcal{E}$-valued deterministic automata and $\mathcal{E}$-valued nondeterministic automata with empty moves. Further, we give the definitions of languages recognized by automata in depth-first and width-first modes, respectively. In Section 3, we re-examine the relation between $\mathcal{E}$-valued deterministic finite-state automata and $\mathcal{E}$-valued nondeterministic finite-state automata. As a corollary, we obtain a pumping lemma for $\mathcal{E}$-valued nondeterministic finite-state automata. In Section 4, the equivalence between $\mathcal{E}$-valued nondeterministic automata and their ϵ-contractions are discussed. In Section 5, we check the closure properties of languages, including disjoint sum, concatenation, homomorphic image, complementation, etc. In Section 6, the generalized Kleene theorem of $\mathcal{E}$-valued nondeterministic automata is discussed. In Section 7, $\mathcal{E}$-valued context-free languages are studied and their generalizations of Chomsky normal form and Greibach normal form are established. In Section 8, we introduce the notion of $\mathcal{E}$-valued pushdown automata and discuss its final state and empty stack acceptance variants in detail. In Section 9, the equivalence between $\mathcal{E}$-valued context-free languages and $\mathcal{E}$-valued pushdown automata is rebuilt.

2. Preliminaries

To begin with, we represent some notions and results in unsharp quantum logic.

Definition 2.1 (*[14]*). A supplement algebra (S-algebra for short) is an algebraic structure $\mathcal{E} = (E, \boxplus, ', \mathbf{0}, \mathbf{1})$ consisting of set M with two constant elements $\mathbf{0}$, $\mathbf{1}$, a unary operation $'$ and a binary operation $\boxplus$ on M satisfying the following axioms:

(S1) $a \boxplus b = b \boxplus a$.
(S2) $a \boxplus (b \boxplus c) = (a \boxplus b) \boxplus c$.
(S3) $a \boxplus a' = \mathbf{1}$.
(S4) $a \boxplus \mathbf{0} = a$.
(S5) $a'' = a$.
(S6) $a \boxplus \mathbf{1} = \mathbf{1}$.

A multiple-valued (MV) algebra [5] is an S-algebra that further satisfies:

(MV) $(a' \boxplus b)' \boxplus b = (a \boxplus b')' \boxplus a$

For an S-algebra, define the following three binary operations:

$$a \odot b = (a' \boxplus b')'$$
$$a \sqcap b = (a \boxplus b') \odot b$$
$$a \sqcup b = (a \odot b') \boxplus b$$

A quantum MV (QMV) algebra [12] is an S-algebra that satisfies:

(QMV1) $a \sqcup (b \sqcap a) = a$.
(QMV2) $(a \sqcap b) \sqcap c = (a \sqcap b) \sqcap (b \sqcap c)$.
(QMV3) $a \boxplus [b \sqcap (a \boxplus c)'] = (a \boxplus b) \sqcap (a \boxplus (a \boxplus c)']$.
(QMV4) $a \boxplus (a' \sqcap b) = a \boxplus b$.
(QMV5) $(a' \boxplus b) \sqcup (b' \boxplus a) = \mathbf{1}$.

A partial relation $\leq$ in QMV algebra is defined as $a \leq b$ iff $a = a \sqcap b$.

It is known that a QMV algebra $\mathcal{E}$ is not necessary a lattice under the operations $\sqcap$ and $\sqcup$ [12]. If $\mathcal{E}$ forms a lattice with $\leq$, it is called a lattice ordered QMV algebra. A QMV-algebra M is quasilinear if $a \not\leq b$ implies $a \sqcap b = b$ [12]. A QMV-algebra (or an MV-algebra) M is linear if $\forall a, b \in M$, either $a \leq b$ or $b \leq a$. There exists a QMV algebra which is not quasilinear (Example 1, [14]). Every MV algebra is a QMV algebra, but there exists a QMV algebra which is not an MV algebra (Example 2.7, [37]).

An effect algebra [9] is a set P with two particular elements $0, 1$ $(0 \neq 1)$, and with a partial binary operation $\oplus : P \times P \longrightarrow P$ such that for all $a, b, c \in P$:

(E1) If $a \oplus b \in P$, then $b \oplus a \in P$ and $a \oplus b = b \oplus a$.
(E2) If $b \oplus c \in P$ and $a \oplus (b \oplus c) \in P$, then $a \oplus b \in P$ and $(a \oplus b) \oplus c \in P$, and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.
(E3) For any $a \in P$ there is a unique $b \in P$ such that $a \oplus b$ is defined, and $a \oplus b = 1$.
(E4) If $1 \oplus a$ is defined, then $a = 0$.

Example 2.1. Let $\varphi = (E, \oplus, 0, 1)$ be an effect algebra. The operation $\oplus$ could be extended to a total operation $\boxplus : E \times E \longrightarrow E$ by defining

$$a \boxplus b = \begin{cases} a \oplus b, & \text{if } (a \oplus b) \text{ is defined} \\ 1, & \text{otherwise} \end{cases}$$

We denote the resulting structure by $\bar{\varphi} = (E, 0, 1, \boxplus)$ and call it an extended effect algebra. From the work [14], we can see that an extended effect algebra $\bar{\varphi}$ preserves the order of the effect algebra and it is equivalent to a quasilinear QMV algebra.

Theorem 2.1 (*[37]*). *Let $\mathcal{E} = (E, \boxplus, ', \mathbf{0}, \mathbf{1})$ be a lattice ordered QMV algebra. The following conditions are equivalent:*

(i) *$\mathcal{E}$ is an MV algebra.*
(ii) *$(a \boxplus b) \wedge (a \boxplus c) = a \boxplus (b \wedge c)$ for any $a, b, c \in E$.*

Theorem 2.2 *([37]). Let $\mathcal{E} = (E, \boxplus, ', \mathbf{0}, \mathbf{1})$ be an extended lattice ordered effect algebra. The following conditions are equivalent:*

(i) *$\mathcal{E}$ is a linear MV algebra.*
(ii) *$(a \boxplus b) \wedge (a \boxplus c) = a \boxplus (b \wedge c)$ for any $a, b, c \in E$.*

As well known, every orthomodular lattice is a special effect algebra by taking *sup* as the $\oplus$. In this effect algebra, all elements are sharp elements (a is sharp if and only if $a \wedge a' = 0$). Certainly it is possible for a general effect algebra that there are also unsharp elements. On the other hand, every effect algebra contains an orthomodular lattice as a subclass, since the set of all sharp elements of an effect algebra determines an orthomodular lattice [8]. Similarly, every orthomodular lattice is also a special QMV algebra by taking *sup* as the $\boxplus$ and $'$ as the orthocomplement. For this QMV algebra, all its elements are idempotent (a is idempotent if and only if $a \boxplus a = a$). Obviously, for a general QMV algebra, there may be elements which are not idempotent. On the other hand, the set of all idempotent elements of a QMV algebra determines an orthomodular lattice [12].

In order to set up an automata theory based on unsharp quantum logic, we represent some notions of the classical automata theory. An automaton is a quintuple $\mathcal{R} = \langle Q, \Sigma, I, T, E\rangle$, where

(i) Q is a finite nonempty set of states.
(ii) Σ is a finite alphabet whose elements are called labels.
(iii) $I \subseteq Q$ is the set of initial states.
(iv) $T \subseteq Q$ is the set of terminal states.
(v) $E \subseteq Q \times \Sigma \times Q$, and each $(p, \sigma, q) \in E$ is called a transition in $\mathcal{R}$, intuitively meaning that input σ causes the automaton move from state p into state q.

Obviously, the conditions (iii), (iv) and (v) in the above definition can be treated as the following propositions with 'yes/no' being their truth value: (a) for any $q \in Q$, is q an initial state? (b) for any $p \in Q$, is p a terminal state? and (c) will the automaton move from p to q when σ is the input? Thus, that the classical automata theory is indeed based on boolean logic [37].

In a similar way, if the truth value set of propositions is described by an unsharp quantum logic, we can develop an automata theory based on unsharp quantum logic. In the following, $\mathcal{E}$ denotes a lattice ordered QMV algebra. If $\mathcal{E}$ denotes an extended lattice ordered effect algebra (or a lattice ordered quasilinear QMV algebra), we can obtain automata theory based on extended lattice ordered effect algebra without changing any further features.

Definition 2.2. An $\mathcal{E}$-valued nondeterministic finite state automaton ($\mathcal{E}$NFA) is a quintuple $M = (Q, \Sigma, I, T, \delta)$, where

(i) Q is a finite nonempty state set.
(ii) Σ is a finite alphabet.
(iii) $I: Q \longrightarrow \mathcal{E}$ is the initial state function.
(iv) $T: Q \longrightarrow \mathcal{E}$ is the terminal state function.
(v) $\delta: Q \times \Sigma_\epsilon \times Q \longrightarrow \mathcal{E}$ is the transition function, where

$$\delta(p, \epsilon, q) = \begin{cases} \mathbf{0}, & \text{if } p = q \\ \mathbf{1}, & \text{if } p \neq q, \end{cases}$$

and $\Sigma_\epsilon = \Sigma \cup \{\epsilon\}$. Similarly to the classical case, $\delta(p, \sigma, q)$ indicates the truth value of the proposition that input σ makes the automaton to move from state p to state q.

For any alphabet Σ, let Σ^* be the free monoid generated by Σ as usual, taking ϵ as the identity (empty) element. Let $\mathcal{H}_\Sigma = \{\sigma_1 \cdots \sigma_n | \sigma_i \in \Sigma_\epsilon, n \geq 1\}$ be the free semigroup generated by Σ_ϵ. The difference between the notions is that $x\epsilon = \epsilon x = x$ in Σ^* since ϵ is the identity element of Σ^*, but $x\epsilon \neq \epsilon x \neq x$ in $\mathcal{H}_\Sigma$. The length function defined on $\mathcal{H}_\Sigma$ is a map $|\cdot| : \mathcal{H}_\Sigma \longrightarrow \mathbf{N}$ such that $|s|$ is the number of all nonempty symbols in s. The virtual length function defined on $\mathcal{H}_\Sigma$ is a map $|\cdot|_v : \mathcal{H}_\Sigma \longrightarrow \mathbf{Z}^+$ such that $|s|_v$ is the number of all empty and nonempty symbols in s. Obviously $|s|_v \geq |s|$, $|s|_v = |s|$ iff $s \in \Sigma^+$. For any alphabet Σ, the ϵ-reduce map is a surjective homomorphism $||\cdot|| : \mathcal{H}_\Sigma \longrightarrow \Sigma^*$ such that $||s||$ is the sequence composed of all nonempty characters of s in their original order if $|s| \geq 1$; define $||s|| = \epsilon$ if $|s| = 0$.

Definition 2.3. An $\mathcal{E}$-valued deterministic finite state automaton ($\mathcal{E}$DFA) is an $\mathcal{E}$NFA whose transform function δ satisfies that, for any $p \in Q$ and any $\sigma \in \Sigma$ there exists at most one $q \in Q$ with $\delta(p, \sigma, q) \neq \mathbf{1}$.

Definition 2.4. An $\mathcal{E}$-valued automaton with empty moves (an $\mathcal{E}\epsilon$NFA) is a quintuple $M = (Q, \Sigma, I, T, \delta)$ satisfying (i)–(iv) of Definition 2.2 and

(vi) $\delta : Q \times \Sigma_\epsilon \times Q \longrightarrow \mathcal{E}$ is the transition function, where $\delta(p, \epsilon, q) = \mathbf{0}$ if $p = q$.

The classes of all $\mathcal{E}$DFA, all $\mathcal{E}$NFA and all $\mathcal{E}\epsilon$NFA over the alphabet Σ are denoted by DFA($\mathcal{E}, \Sigma$), NFA($\mathcal{E}, \Sigma$) and ϵNFA($\mathcal{E}, \Sigma$) respectively.

Definition 2.5. An $\mathcal{E}$-valued language L on Σ is a map $L : \Sigma^* \longrightarrow \mathcal{E}$.

Definition 2.6. Let f, g be $\mathcal{E}$-valued languages.

(i) The intersection of two $\mathcal{E}$-valued languages f and g, denoted by $f \wedge g$, is defined as $(f \wedge g)(s) = f(s) \wedge g(s)$ for any $s \in \Sigma^*$.
(ii) The sum of $\mathcal{E}$-valued languages f and g, denoted by $f \boxplus g$, is defined as $(f \boxplus g)(s) = f(s) \boxplus g(s)$ for any $s \in \Sigma^*$.
(iii) Denote $s^R = \sigma_n \cdots \sigma_1$ for any $s = \sigma_1 \cdots \sigma_n \in \Sigma^n$ $(n \geq 1)$, and $\epsilon^R = \epsilon$. The reversal of an $\mathcal{E}$-valued language f is defined as $f^R(s) = f(s^R)$.
(iv) The concatenation of two $\mathcal{E}$-valued languages f and g, denoted by $f \cdot g$, is defined as $(f \cdot g)(s) = \bigwedge_{s_1 s_2 = s}[f(s_1) \boxplus g(s_2)]$ for any $s \in \Sigma^*$.

As in the automata theory based on orthomodular lattice, by using $\wedge$ and $\boxplus$, we can adapt the depth-first and width-first modes respectively to define the acceptance degree of language recognized by an automaton. It is shown that only when the truth-lattice is an MV algebra, the two ways coincide.

Definition 2.7 (*n-path*). An n-path π between states p and q in M is a finite sequence of states $\pi = (p_0 = p, p_1, p_2, \ldots, p_n = q)$. In a given $\mathcal{E}$-automaton M, the set of all paths $\pi = (p_0 = p, p_1, p_2, \ldots, p_n = q)$ of length n between p and q will be denoted by $P_M^n(p, q)$.

The n-path π is assigned with the function $||\pi|| : \mathcal{H}_\Sigma \longrightarrow \mathcal{E}$, such that

$$||\pi||(\sigma_1 \cdots \sigma_n) = \boxplus_{i=0,1,\ldots,n-1} \delta(p_i, \sigma_{i+1}, p_{i+1})$$

Definition 2.8. The $\mathcal{E}$-valued language accepted by an $\mathcal{E}$NFA M in a depth-first mode, denoted by $|M|_d$, is defined as

$$\begin{aligned} |M|_d(s) &= \bigwedge_{p,q \in Q} \bigwedge_{\pi \in P_M^n(p,q)} I(p) \boxplus ||\pi||(s) \boxplus T(q) \\ &= \bigwedge_{p_i \in Q} I(p_0) \boxplus \delta(p_0, \sigma_1, p_1) \boxplus \cdots \boxplus \delta(p_{n-1}, \sigma_n, p_n) \boxplus T(p_n) \end{aligned}$$

where $s = \sigma_1 \cdots \sigma_n \in \Sigma^*$.

Definition 2.9. The $\mathcal{E}$-valued language accepted by an $\mathcal{E}\epsilon$NFA M in a depth-first mode, denoted by $|M|_d$, is defined as

$$\begin{aligned} |M|_d(s) &= \bigwedge_{||s'||=s} \bigwedge_{p,q \in Q} \bigwedge_{\pi \in P_M^n(p,q)} I(p) \boxplus ||\pi||(s') \boxplus T(q) \\ &= \bigwedge_{||s'||=s} \bigwedge_{p_i \in Q} I(p_0) \boxplus \delta(p_0, \sigma'_1, p_1) \boxplus \cdots \boxplus \delta(p_{m-1}, \sigma'_m, p_m) \boxplus T(p_m) \end{aligned} \tag{1}$$

where $s = \sigma_1 \cdots \sigma_n \in \Sigma^*$ and $s' = \sigma'_1 \cdots \sigma'_m \in \mathcal{H}_\Sigma$.

In the following, we will find that any path containing sufficient ϵ-moves can be disregarded in the final $\wedge$-process, since the value of such path must be greater than the value of some path containing less ϵ-moves. As a result, we can simplify Eq. (1) by only calculating these s' whose virtual length is less than some upper bound.

Proposition 2.3. *For any $\mathcal{E}\epsilon$NFA $M = (Q, \Sigma, I, T, \delta)$ and every $s = \sigma_1 \cdots \sigma_n \in \Sigma^*$, there exists some positive integer $N = N(M, n)$ such that*

$$|M|_d(s) = \bigwedge_{||s'||=s, |s'|_v \leq N} \left(\bigwedge_{p_i \in Q, i \leq m} I(p_0) \boxplus \delta(p_0, \sigma'_1, p_1) \boxplus \cdots \boxplus \delta(p_{m-1}, \sigma'_m, p_m) \boxplus T(p_m) \right) \tag{2}$$

where $m = |s'|_v$.

Proof. Suppose $s' = \sigma'_1 \cdots \sigma'_m$ satisfying $||s'|| = s$ and $\{p_i\}_{i=0}^m$ is a list of states. Assume that $\sigma'_i = \sigma_u, \sigma'_j = \sigma_{u+1}$, then $\sigma'_{i+1} = \cdots = \sigma'_{j-1} = \epsilon$. There must be $p_k = p_l, i \leq k < l \leq j$ if $j - i > |Q|$. Thus $\delta(p_0, \sigma'_1, p_1) \boxplus \cdots \boxplus \delta(p_{m-1}, \sigma'_m, p_m) \geq \delta(p_0, \sigma'_1, p_1) \boxplus \cdots \boxplus \delta(p_{k-1}, \sigma'_k, p_k) \boxplus \delta(p_l, \sigma'_{l+1}, p_{l+1}) \boxplus \cdots \boxplus \delta(p_{m-1}, \sigma'_m, p_m)$, where $s'' = \sigma'_1 \cdots \sigma'_k \sigma'_{l+1} \cdots \sigma'_m$. Note that $|s'|_v > |s''|_v$, by iterating the above process we get that

$$\delta(p_0, \sigma'_1, p_1) \boxplus \cdots \boxplus \delta(p_{m-1}, \sigma'_m, p_m) \geq \delta(q_0, \omega_1, q_1) \boxplus \cdots \boxplus \delta(q_{r-1}, \omega_r, q_r)$$

where $\omega_1 \cdots \omega_r$ is a subsequence of s', $\{q_j\}_{j=1}^r$ is a subsequence of $\{p_i\}_{i=1}^m$ and $r \leq n|Q| + |Q| - 1$. Denote $N = n|Q| + |Q| - 1$, that is the conclusion. $\square$

Now we define another way for an $\mathcal{E}\epsilon$NFA to recognize language. Let $M = (Q, \Sigma, I, T, \delta)$ be an $\mathcal{E}\epsilon$NFA, define $\delta_w : \mathcal{E}^Q \times \Sigma_\epsilon \longrightarrow \mathcal{E}^Q$ as

$$\delta_w(X, \sigma)(q) = \bigwedge_{p \in Q} X(p) \boxplus \delta(p, \sigma, q) \tag{3}$$

for any $X \in \mathcal{E}^Q$. Define $\delta_w(X, t\sigma) = \delta_w(\delta_w(X, t), \sigma)$ for any $t \in \mathcal{H}_\Sigma, \sigma \in \Sigma_\epsilon$.

Given $X \in \mathcal{E}^Q$, we simply write $X = \mathbf{0}(\mathbf{1})$ if $X(p) = \mathbf{0}(\mathbf{1})$ for all $p \in Q$.

Remark 2.1. It follows from the definition that if $\delta_w(X, t) = \mathbf{1}$, then $\delta_w(X, ts) = \mathbf{1}$ for any $s \in \mathcal{H}_\Sigma$.

Definition 2.10. The $\mathcal{E}$-valued language accepted by an $\mathcal{E}\epsilon$NFA $M = (Q, \Sigma, I, T, \delta)$ in a width-first mode, denoted by $|M|_w$, is defined as

$$|M|_w(s) = \bigwedge_{||t||=s, t\in\mathcal{H}_\Sigma} \left(\bigwedge_{q\in Q} \delta_w(I, t)(q) \boxplus T(q) \right) \tag{4}$$

for any $s \in \Sigma^*$.

If M is an $\mathcal{E}$NFA, Eq. (4) is

$$|M|_w(s) = \bigwedge_{q\in Q} \delta_w(I, s)(q) \boxplus T(q) \tag{5}$$

Note that $\delta_w(X, \epsilon) = X$ when $M \in$NFA$(\mathcal{E}, \Sigma)$. Write Eq. (5) in a more computable form:

$$|M|_w(s) = \bigwedge_{p_n} \left(\cdots \left(\bigwedge_{p_1} \left(\bigwedge_{p_0} I(p_0) \boxplus \delta(p_0, \sigma_1, p_1) \right) \boxplus \delta(p_1, \sigma_2, p_2) \right) \cdots \right) \boxplus T(p_n) \tag{6}$$

for any $s = \sigma_1 \cdots \sigma_n \in \Sigma^+$. If $s = \epsilon$, $|M|_w(\epsilon) = \wedge_{q\in Q} \delta_w(I, \epsilon)(q) \boxplus T(q) = \wedge_q I(q) \boxplus T(q) = |M|_d(\epsilon)$.

Definition 2.11. Denote $L_d(\mathcal{E}, \Sigma) = \{|M|_d \in \mathcal{E}^{\Sigma^*} : M \in \text{NFA}(\mathcal{E}, \Sigma)\}$ and $L_w(\mathcal{E}, \Sigma) = \{|M|_w \in \mathcal{E}^{\Sigma^*} : M \in \text{NFA}(\mathcal{E}, \Sigma)\}$. Language L is called d-regular language or w-regular language if $L \in L_d(\mathcal{E}, \Sigma)$ or $L_w(\mathcal{E}, \Sigma)$.

In the above discussion, similar to [26], we defined the $\mathcal{E}$-valued languages in depth-first and width-first mode according to the way the acceptance degree is calculated. In depth-first mode, we first calculated the acceptance degree of s for each single path, then unite them together. In width-first mode, we take the first transition of all paths and unite their acceptance degrees together by *inf* operation. And then we take the second, third, . . . transition of all paths and unite their acceptance degrees separately. At last, we perform the $\boxplus$ operation on all these united values and get the wanted general acceptance degree of s.

If M is an $\mathcal{E}\epsilon$NFA, it is also said to recognize L in a depth-first mode if $L = |M|_d$ or called recognize L in a width-first way if $L = |M|_w$.

Proposition 2.4. (i) *$|M|_w \le |M|_d$ for any $M \in \epsilon$NFA$(\mathcal{E}, \Sigma)$.*
(ii) *$|M|_w = |M|_d$ for any $M \in \epsilon$NFA$(\mathcal{E}, \Sigma)$ iff $\mathcal{E}$ is an MV algebra.*

Proof. (i) Generally in $\mathcal{E}$ there is $a \boxplus (b \wedge c) \le (a \boxplus b) \wedge (a \boxplus c)$. For any $s \in \Sigma^*$,

$$\begin{aligned} |M|_w(s) &= \wedge_{||t||=s} \wedge_{p_n} \left(\wedge_{p_{n-1}} \left(\cdots \left(\wedge_{p_0} I(p_0) \boxplus \delta(p_0, \sigma_1, p_1) \right) \boxplus \cdots \right) \boxplus \delta(p_{n-1}, \sigma_n, p_n) \right) \boxplus T(p_n) \\ &\le \wedge_{||t||=s} \wedge_{p_i\in Q} I(p_0) \boxplus \delta(p_0, \sigma_1, p_1) \boxplus \cdots \boxplus \delta(p_{n-1}, \sigma_n, p_n) \boxplus T(p_n) \\ &= |M|_d(s) \end{aligned}$$

(ii) If $\mathcal{E}$ is an MV algebra then $\boxplus$ distributes over $\wedge$, so $|M|_w = |M|_d$. Conversely, for any $a, b, c \in E$, construct an $\mathcal{E}$NFA $M = (\{q_0, q_1, q_2, q_3\}, \{\sigma\}, I, T, \delta)$ as follows:

$$\begin{aligned} &I(q_0) = I(q_1) = \mathbf{0}, I(q_2) = I(q_3) = \mathbf{1} \\ &T(q_0) = T(q_1) = T(q_2) = \mathbf{1}, T(q_3) = \mathbf{0} \\ &\delta(q_0, \sigma, q_2) = b, \delta(q_1, \sigma, q_2) = c, \delta(q_2, \sigma, q_3) = a \end{aligned}$$

and $\delta(p, \sigma, q) = \mathbf{1}$ for other (p, q) pairs. Thus $|M|_d(\sigma\sigma) = (b \boxplus a) \wedge (c \boxplus a) = |M|_w(\sigma\sigma) = (b \wedge c) \boxplus a$, and $\mathcal{E}$ is an MV algebra. $\square$

Remark 2.2. The domain of transition function δ could be extended to $\mathcal{H}_\Sigma$: define

$$\delta^*(p, s, q) = \bigwedge_{p_1,\ldots,p_{n-1}\in Q} \delta(p, \sigma_1, p_1) \boxplus \cdots \boxplus \delta(p_{n-1}, \sigma_n, q)$$

for any $s = \sigma_1 \cdots \sigma_n \in \mathcal{H}_\Sigma$. In the following, we still write δ^* as δ.

3. $\mathscr{E}$DFA and $\mathscr{E}$NFA

In [37] (using the depth-first mode), we have proved that an $\mathscr{E}$-valued nondeterministic finite state automaton and its subset construction have the same ability of accepting language, if and only if the underlying lattice degenerates into an MV algebra. However, the next theorem indicates that any $\mathscr{E}$NFA can be simulated by its power set construction within width-first principle.

Denote the range of a map f to be $R(f)$. For an $\mathscr{E}$NFA $M = (Q, \Sigma, I, T, E)$ denote $R_M = R(I) \cup R(E) \cup R(T)$. Let S_M denote the subalgebra in $\mathscr{E}$ generated by R_M, then S_M^Q is a subset of $\mathscr{E}^Q$.

Theorem 3.1. *Let M be an $\mathscr{E}$NFA. There exists an $\mathscr{E}$DFA $\bar{M}$ such that $|\bar{M}|_d = |M|_w$ if S_M is finite.*

Proof. Assume $M = (Q, \Sigma, I, T, \delta)$. S_M^Q is finite by hypothesis. Construct an $\mathscr{E}$DFA $\bar{M} = (S_M^Q, \Sigma, \bar{I}, \bar{T}, \bar{\delta})$ as following:

$$\bar{I}(X) = \begin{cases} \mathbf{0}, & \text{if } X = I \\ \mathbf{1}, & \text{otherwise} \end{cases}$$

$$\bar{T}(X) = \wedge_{p\in Q} (X(p) \boxplus T(p))$$

$\bar{\delta}$ is the restriction of δ_w on S_M^Q. Thus $|\bar{M}|_d = |M|_w$ by definition. $\square$

In the classical automata theory, the pumping lemma is a very useful tool to show that a language is not regular. In general, the pumping lemma is not true for $\mathscr{E}$-valued language. However, when using the width-first acceptance, we can prove the pumping lemma for $\mathscr{E}$-valued languages.

Lemma 3.2 (*Pumping Lemma*)**.** *Let $M = (Q, \Sigma, I, T, \delta)$ be an $\mathscr{E}$NFA and N be the cardinal of S_M^Q. If N is finite, for any $s \in \Sigma^+$, if $|s| > N$ then it can be divided as $s = uvw$ such that $|M|_w(s) = |M|_w(uv^iw)$ where $u, w \in \Sigma^*$, $v \in \Sigma^+$ and $i \in \mathbf{N}$.*

Proof. Suppose $\bar{M} = (S_M^Q, \Sigma, \bar{I}, \bar{T}, \bar{\delta})$ is the $\mathscr{E}$DFA constructed from M stated in Theorem 3.1. Denote $s = \sigma_1 \cdots \sigma_n \in \Sigma^+$ and the only nontrivial path of $\bar{M}$ recognizing s is $(X_0 = I, X_1, \ldots, X_n)$. Then $|\bar{M}|_d(s) = \bar{T}(X_n)$. When $n > N$, there must be $X_j = X_k$ for some $0 \le j < k \le n$. Let $v = \sigma_{j+1} \cdots \sigma_k,\ u = \sigma_1 \cdots \sigma_j,\ w = \sigma_{k+1} \cdots \sigma_n$. It is easy to see that the terminal state of the path accepting uv^iw in $\bar{M}$ is always X_n. So $|M|_w(uv^iw) = |\bar{M}|_d(uv^iw) = \bar{T}(X_n) = |\bar{M}|_d(s) = |M|_w(s)$. $\square$

4. $\mathscr{E}$NFA and $\mathscr{E}\epsilon$NFA

In the classical automata theory, automata with ϵ-moves are very useful in proving the closures properties of regular languages and have the same acceptance power as the nondeterministic finite-state automata. In this section, we study $\mathscr{E}\epsilon$NFA and prove that $\mathscr{E}\epsilon$NFA and its ϵ-contraction have the same recognition power if and only if the underlying truth valued lattice is an MV algebra.

Definition 4.1. For any $\mathscr{E}\epsilon$NFA $M = (Q, \Sigma, I, T, \delta)$, define the reduction of M to be an $\mathscr{E}$NFA $M^c = (Q, \Sigma, I, T^c, \delta^c)$:

$$\delta^c(p, \sigma, q) = \bigwedge_{k\ge 0, p'\in Q} \left[\delta(p, \epsilon^k, p') \boxplus \delta(p', \sigma, q)\right]$$

$$T^c(p) = \bigwedge_{k\ge 0, p'\in Q} \left[\delta(p, \epsilon^k, p') \boxplus T(p')\right]$$

where $\sigma \in \Sigma$, $\delta(p, \epsilon^k, q) = \bigwedge_{p_i\in Q} [\delta(p, \epsilon, p_1) \boxplus \cdots \boxplus \delta(p_{k-1}, \epsilon, q)]$ for $k \ge 2$ and $\delta(p, \epsilon^0, q) = \mathbf{0}$, $\delta(p, \epsilon^1, q) = \delta(p, \epsilon, q)$. And $\delta^c(p, \epsilon, q) = \mathbf{1}$ if $p \ne q$.

Theorem 4.1. *For any $\mathscr{E}\epsilon$NFA M and the $\mathscr{E}$NFA M^c constructed as above,*

(i) $|M|_d = |M^c|_d$ *iff* $(a \boxplus c) \wedge (b \boxplus c) = (a \wedge b) \boxplus c$ *for any* $a, b, c \in \mathscr{E}$.
(ii) $|M|_w = |M^c|_w$ *iff* $(a \boxplus c) \wedge (b \boxplus c) = (a \wedge b) \boxplus c$ *for any* $a, b, c \in \mathscr{E}$.

Proof. (i) "If part". Denote $M = (Q, \Sigma, I, T, \delta)$ and $\mathscr{E}$NFA $M^c = (Q, \Sigma, I, T^c, \delta^c)$.
First, we prove that

$$T^c(p) = \bigwedge_{k\le |Q|, p'\in Q} \left[\delta(p, \epsilon^k, p') \boxplus T(p')\right]$$

$$\delta^c(p, \sigma, q) = \bigwedge_{k\le |Q|, p'\in Q} \left[\delta(p, \epsilon^k, p') \boxplus \delta(p', \sigma, q)\right]$$

For any given set of states $\{p_1, \ldots, p_k\}$, if $k > |Q|$, then there must be $p_i = p_j$ for some $1 \le i < j \le k$ since Q is finite. Thus, $\delta(p_0, \epsilon, p_1) \boxplus \cdots \boxplus \delta(p_{k-1}, \epsilon, p_k) \ge \delta(p_0, \epsilon, p_1) \boxplus \cdots \boxplus \delta(p_{i-1}, \epsilon, p_i) \boxplus \delta(p_j, \epsilon, p_{j+1}) \boxplus \cdots \boxplus \delta(p_{k-1}, \epsilon, p_k)$ by the monotonicity of $\boxplus$. So for any $k \ge |Q|$, $l(k) \le |Q|$,

$$\delta(p, \epsilon^k, q) = \bigwedge_{p_i\in Q} [\delta(p, \epsilon, p_1) \boxplus \cdots \boxplus \delta(p_{k-1}, \epsilon, q)] \ge \bigwedge_{p_i\in Q} \left[\delta(p, \epsilon, p_1) \boxplus \cdots \boxplus \delta(p_{l(k)-1}, \epsilon, q)\right] = \delta(p, \epsilon^{l(k)}, q)$$

Then

$$\begin{aligned}
T^c(p) &= \bigwedge_{k\geq 0, p'\in Q} \left[\delta(p,\epsilon^k,p') \boxplus T(p')\right] \\
&= \bigwedge_{k\geq |Q|, p'\in Q} \left[\delta(p,\epsilon^k,p') \boxplus T(p')\right] \wedge \bigwedge_{k\leq |Q|, p'\in Q} \left[\delta(p,\epsilon^k,p') \boxplus T(p')\right] \\
&\geq \bigwedge_{l(k)\leq |Q|, p'\in Q} \left[\delta(p,\epsilon^{l(k)},p') \boxplus T(p')\right] \wedge \bigwedge_{k\leq |Q|, p'\in Q} \left[\delta(p,\epsilon^k,p') \boxplus T(p')\right] \\
&= \bigwedge_{k\leq |Q|, p'\in Q} \left[\delta(p,\epsilon^k,p') \boxplus T(p')\right]
\end{aligned}$$

Obviously, $T^c(p) \leq \bigwedge_{k\leq |Q|, p'\in Q} \left[\delta(p,\epsilon^k,p') \boxplus T(p')\right]$ by Definition 4.1, hence, $T^c(p) = \bigwedge_{k\leq |Q|, p'\in Q} \left[\delta(p,\epsilon^k,p') \boxplus T(p')\right]$. Similarly, $\delta^c(p,\sigma,q) = \bigwedge_{k\leq |Q|, p'\in Q} \left[\delta(p,\epsilon^k,p') \boxplus \delta(p',\sigma,q)\right]$.

Since $||s'|| = ||\sigma'_1\cdots\sigma'_m|| = \sigma_1\cdots\sigma_n$, denote $\sigma'_{m_i} = \sigma_i$ $(i = 1,\ldots,n)$, $m_0 = 0, m_{n+1} = m$. Define $U = \{(0,m_1,\ldots,m_{n+1}) | m_{i+1} - m_i \leq |Q| + 1, m_{n+1} - m_n \leq |Q|, i = 0,\ldots,n-1\} \subset \{0\} \times \mathbb{N}^{n+1}$, $V = \{0\} \times \mathbb{N}^{n+1} - U$. For any $s = \sigma_1\sigma_2\cdots\sigma_n$, the value of s accepted by M is

$$\begin{aligned}
|M|_d(s) &= \bigwedge_{||s'||=s} \bigwedge_{p_j} (I(p_0) \boxplus \delta(p_0,\sigma'_1,p_1) \boxplus \cdots \boxplus \delta(p_{m-1},\sigma'_m,p_m) \boxplus T(p_m)) \\
&= \bigwedge_{m_i<m_{i+1}} \bigwedge_{p_j, j=0,\ldots,m} (I(p_0) \boxplus \delta(p_0,\epsilon,p_1) \boxplus \cdots \boxplus \delta(p_{m_1-1},\sigma_1,p_{m_1}) \\
&\quad \boxplus \cdots \boxplus \delta(p_{m_n-1},\sigma_n,p_{m_n}) \boxplus \delta(p_{m_n},\epsilon,p_{m_n+1}) \boxplus \cdots \boxplus T(p_m)) \\
&= \bigwedge_{m_i<m_{i+1}} \bigwedge_{p_{m_i}\in Q} (I(p_0) \boxplus \sum_{i=0}^{n-1} \left[\bigwedge_{q_{m_i}\in Q} \delta(p_{m_i},\epsilon^{m_{i+1}-m_i-1},q_{m_i}) \boxplus \delta(q_{m_i},\sigma_{m_{i+1}},p_{m_{i+1}}) \right] \\
&\quad \boxplus \delta(p_{m_n},\epsilon^{m-m_n},p_m) \boxplus T(p_m)) \\
&= \bigwedge_{U\cup V} \bigwedge_{p_{m_i}\in Q} (I(p_0) \boxplus \sum_{i=0}^{n-1} \left[\bigwedge_{q_{m_i}\in Q} \delta(p_{m_i},\epsilon^{m_{i+1}-m_i-1},q_{m_i}) \boxplus \delta(q_{m_i},\sigma_{m_{i+1}},p_{m_{i+1}}) \right] \\
&\quad \boxplus \delta(p_{m_n},\epsilon^{m-m_n},p_m) \boxplus T(p_m))
\end{aligned} \tag{7}$$

For any $(m_0,\ldots,m_{n+1}) \in V$, there is $m_{j+1} - m_j - 1 > |Q|$ for some $0 \leq j < n$ or $m_{n+1} - m_n > |Q|$. Then

$$\begin{aligned}
&I(p_0) \boxplus \sum_{i=0}^{n-1} \left[\bigwedge_{q_{m_i}\in Q} \delta(p_{m_i},\epsilon^{m_{i+1}-m_i-1},q_{m_i}) \boxplus \delta(q_{m_i},\sigma_{m_{i+1}},p_{m_{i+1}}) \right] \boxplus \delta(p_{m_n},\epsilon^{m-m_n},p_m) \boxplus T(p_m) \\
&\geq I(p_0) \boxplus \sum_{i=0}^{j-1} \left[\bigwedge_{q_{m_i}\in Q} \delta(p_{m_i},\epsilon^{m_{i+1}-m_i-1},q_{m_i}) \boxplus \delta(q_{m_i},\sigma_{m_{i+1}},p_{m_{i+1}}) \right] \boxplus \bigwedge_{l\leq |Q|, q_j\in Q} E(p_{m_j},\epsilon^l,q_j) \\
&\quad \boxplus\ \delta(q_j,\sigma_{m_{j+1}},p_{m_{j+1}}) \boxplus \cdots \boxplus \delta(p_{m_n},\epsilon^{m-m_n},p_m) \boxplus T(p_m)
\end{aligned}$$

Therefore, $\bigwedge_{(m_0,\ldots,m_{n+1})\in V}(\cdots) \geq \bigwedge_{(m_0,\ldots,m_{n+1})\in U}(\cdots)$ in Eq. (7) and

$$\begin{aligned}
|M|_d(s) &= \bigwedge_{U} \bigwedge_{p_{m_i}\in Q} I(p_0) \boxplus \sum_{i=0}^{n-1} \left[\bigwedge_{q_{m_i}\in Q} \delta(p_{m_i},\epsilon^{m_{i+1}-m_i-1},q_{m_i}) \boxplus \delta(q_{m_i},\sigma_{m_{i+1}},p_{m_{i+1}}) \right] \boxplus \delta(p_{m_n},\epsilon^{m-m_n},p_m) \boxplus T(p_m) \\
&= \bigwedge_{p_{m_i}\in Q, i\leq n} I(p_0) \boxplus \sum_{i=0}^{n-1} \left[\bigwedge_{m_{i+1}-m_i-1\leq |Q|} \bigwedge_{q_{m_i}\in Q} \delta(p_{m_i},\epsilon^{m_{i+1}-m_i-1},q_{m_i}) \boxplus \delta(q_{m_i},\sigma_{m_{i+1}},p_{m_{i+1}}) \right]
\end{aligned}$$

$$\boxplus \left[\bigwedge_{m-m_n \le |Q|} \bigwedge_{p_m \in Q} \delta(p_{m_n}, \epsilon^{m-m_n}, p_m) \boxplus T(p_m) \right]$$
$$= \bigwedge_{p_{m_i} \in Q, i \le n} I(p_0) \boxplus \sum_{i=0}^{n-1} \delta^c(p_{m_i}, \sigma_{m_i+1}, p_{m_i+1}) \boxplus T^c(p_{m_n})$$
$$= |M^c|_d(s)$$

In the case of $s = \epsilon$,

$$\begin{aligned} |M|_d(\epsilon) &= \bigwedge_{n \ge 0} \left[\bigwedge_{p_0, p_n \in Q} I(p_0) \boxplus \delta(p_0, \epsilon^n, p_n) \boxplus T(p_n) \right] \\ &= \bigwedge_{0 \le n \le |Q|} \left[\bigwedge_{p_0, p_n \in Q} I(p_0) \boxplus \delta(p_0, \epsilon^n, p_n) \boxplus T(p_n) \right] \\ &= \bigwedge_{p_0 \in Q} I(p_0) \boxplus \left[\bigwedge_{n \le |Q|, p_n \in Q} \delta(p_0, \epsilon^n, p_n) \boxplus T(p_n) \right] \\ &= \bigwedge_{p_0 \in Q} I(p_0) \boxplus T^c(p_0) = |M^c|_d(\epsilon) \end{aligned}$$

So an $\mathcal{E}\epsilon$NFA could be simulated by an $\mathcal{E}$NFA.

"Only if part". For any $a, b, c \in \mathcal{E}$, consider an $\mathcal{E}\epsilon$NFA $M = (Q, \Sigma, I, T, E)$, where $Q = \{p_0, p_1, p_2\}$, $\sigma \in \Sigma$,

$$\begin{aligned} &I(p_0) = \mathbf{0}, I(p_1) = I(p_2) = \mathbf{1} \\ &T(p_2) = \mathbf{0}, T(p_0) = T(p_1) = \mathbf{1} \\ &\delta(p_0, \sigma, p_1) = a, \delta(p_1, \sigma, p_1) = b, \delta(p_1, \epsilon, p_2) = c \\ &\delta(p_0, \epsilon, p_1) = \delta(p_2, \epsilon, p_2) = \mathbf{0} \end{aligned}$$

and all other values of δ are $\mathbf{1}$. By the definition of M^c,

$$\begin{aligned} &\delta^c(p_0, \sigma, p_1) = \delta(p_0, \sigma, p_1) \wedge [\delta(p_0, \epsilon, p_1) \boxplus \delta(p_1, \sigma, p_1)] = a \wedge b \\ &\delta^c(p_1, \sigma, p_2) = \delta^c(p_0, \sigma, p_2) = \mathbf{1} \end{aligned}$$

By the definition of T^c,

$$\begin{aligned} &T^c(p_0) = \mathbf{1} \\ &T^c(p_1) = \delta(p_1, \epsilon, p_2) \boxplus T(p_2) = c \\ &T^c(p_2) = \delta(p_2, \epsilon, p_2) \boxplus T(p_2) = \mathbf{0} \end{aligned}$$

therefore $|M^c|_d(\sigma) = I(p_0) \boxplus \delta^c(p_0, \sigma, p_1) \boxplus T^c(p_1) = (a \wedge b) \boxplus c$. On the other side,

$$\begin{aligned} |M|_d(\sigma) &= [I(p_0) \boxplus \delta(p_0, \sigma, p_1) \boxplus \delta(p_1, \epsilon, p_2) \boxplus \delta(p_2, \epsilon^k, p_2) \boxplus T(p_2)] \\ &\quad \wedge [I(p_0) \boxplus \delta(p_0, \epsilon, p_1) \boxplus \delta(p_1, \sigma, p_1) \boxplus \delta(p_1, \epsilon, p_2) \boxplus T(p_2)] \\ &= (a \boxplus c) \wedge (b \boxplus c) \end{aligned}$$

So $(a \boxplus c) \wedge (b \boxplus c) = (a \wedge b) \boxplus c$ for any $a, b, c \in \mathcal{E}$.

(ii) "If part". If $\boxplus$ distributes over $\wedge$, then $\mathcal{E}$ is an MV algebra. Thus $|M^c|_w = |M^c|_d = |M|_d = |M|_w$ by Proposition 2.4 and (i).

"Only if part". For any $a, b, c \in \mathcal{E}$, construct $M = (\{p_0, p_1, p_2, p_3\}, \Sigma, I, T, \delta)$ as follows: for some $\sigma \in \Sigma$,

$$\begin{aligned} &I(p_0) = I(p_1) = \mathbf{0}, I(p_2) = I(p_3) = \mathbf{1} \\ &T(p_0) = T(p_1) = T(p_2) = \mathbf{1}, T(p_3) = \mathbf{0} \\ &\delta(p_0, \epsilon, p_2) = a, \delta(p_1, \epsilon, p_2) = b, \delta(p_2, \sigma, p_3) = c \end{aligned}$$

and all other values of δ are $\mathbf{1}$. The corresponding $M^c = (\{p_0, p_1, p_2, p_3\}, \Sigma, I, T^c, \delta_c)$ is: $T^c = T$, $\delta^c(p_0, \sigma, p_3) = a \boxplus c$, $\delta^c(p_1, \sigma, p_3) = b \boxplus c$, $\delta^c(p_2, \sigma, p_3) = c$ and all other values of δ^c are $\mathbf{1}$. Then $|M^c|_w(\sigma) = \wedge_{q_1}(\wedge_{q_0} I(q_0) \boxplus \delta^c(q_0, \sigma, q_1)) \boxplus T^c(q_1) = (\delta^c(p_0, \sigma, p_3) \wedge \delta^c(p_1, \sigma, p_3)) \boxplus T^c(p_3) = (a \boxplus c) \wedge (b \boxplus c)$. On the other hand, $|M|_w(\sigma) = [(I(p_0) \boxplus \delta(p_0, \epsilon, p_2) \wedge (I(p_1) \boxplus \delta(p_1, \epsilon, p_2)] \boxplus \delta(p_2, \sigma, p_3) \boxplus T(p_3) = (a \wedge b) \boxplus c$. If $|M^c|_w = |M|_w$, then $(a \boxplus c) \wedge (b \boxplus c) = (a \wedge b) \boxplus c$. $\square$

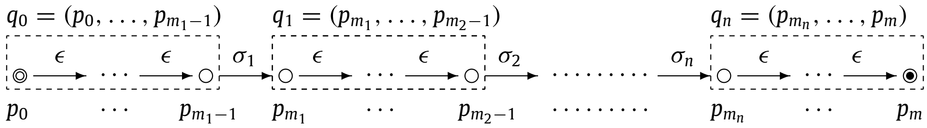

Fig. 1. Single path simulation.

In the following, we show that for any $\mathcal{E}\epsilon$NFA, we can construct a $\mathcal{E}$NFA with the recognition power equal to the original $\mathcal{E}\epsilon$NFA in depth-first mode, and distributivity is not required.

Definition 4.2. For any $\mathcal{E}\epsilon$NFA $M = (Q, \Sigma, I, T, \delta)$, define an $\mathcal{E}$NFA $M^e = (Q^e, \Sigma, I^e, T^e, \delta^e)$ as following:

- $Q^e = Q^1 \cup Q^2 \cup \cdots \cup Q^{|Q|}$ where Q^k is the cartesian product of k-copies of Q.
- $I^e(q) = I(p_1)$, if $q = (p_1, \ldots, p_n)$ where $p_i \in Q$.
- $T^e(q) = \delta(p_1, \epsilon, p_2) \boxplus \cdots \boxplus \delta(p_{n-1}, \epsilon, p_n) \boxplus T(p_n)$, if $q = (p_1, \ldots, p_n)$ where $p_i \in Q$.
- $\delta^e(q_1, \sigma, q_2) = \delta(p_1, \epsilon, p_2) \boxplus \cdots \boxplus \delta(p_{n-1}, \epsilon, p_n) \boxplus \delta(p_n, \sigma, p'_1)$, where $q_1 = (p_1, \ldots, p_n) \in Q^n$ and $q_2 = (p'_1, \ldots, p'_m) \in Q^m$.

Each path of M is simulated by a path of M^e, illustrated as Fig. 1.

Theorem 4.2. *$|M^e|_d = |M|_d$ for any $M \in \epsilon NFA(\mathcal{E}, \Sigma)$.*

Proof. Assume the $\mathcal{E}\epsilon$NFA $M = (Q, \Sigma, I, T, \delta)$. For any $s \in \Sigma^*$,

$$\begin{aligned}
|M^e|_d(s) &= \bigwedge_{q_i \in Q^e} I^e(q_0) \boxplus \delta^e(q_0, \sigma_1, q_1) \boxplus \cdots \boxplus \delta^e(q_{n-1}, \sigma_n, q_n) \boxplus T^e(q_n) \\
&= \bigwedge_{q_i \in Q^e} I(p_1^0) \boxplus [\delta(p_1^0, \epsilon, p_2^0) \boxplus \cdots \boxplus \delta(p_{k_0-1}^0, \epsilon, p_{k_0}^0) \boxplus \delta(p_{k_0}^0, \sigma_1, p_1^1)] \\
&\quad \boxplus \cdots \boxplus [\delta(p_1^{n-1}, \epsilon, p_2^{n-1}) \boxplus \cdots \boxplus \delta(p_{k_{n-1}-1}^{n-1}, \epsilon, p_{k_{n-1}}^{n-1}) \boxplus \delta(p_{k_{n-1}}^{n-1}, \sigma_n, p_1^n)] \\
&\quad \boxplus [\delta(p_1^n, \epsilon, p_2^n) \boxplus \cdots \boxplus \delta(p_{k_n-1}^n, \epsilon, p_{k_n}^n) \boxplus T(p_{k_n}^n)] \\
&= \bigwedge_{p_j \in Q} I(p_1) \boxplus [\delta(p_1, \epsilon, p_2) \boxplus \cdots \boxplus \delta(p_{k_0-1}, \epsilon, p_{k_0}) \boxplus \delta(p_{k_0}, \sigma_1, p_{k_0+1})] \\
&\quad \boxplus \cdots \boxplus [\delta(p_{\alpha_{n-2}+1}, \epsilon, p_{\alpha_{n-2}+2}) \boxplus \cdots \boxplus \delta(p_{\alpha_{n-1}-1}, \epsilon, p_{\alpha_{n-1}}) \boxplus \delta(p_{\alpha_{n-1}}, \sigma_n, p_{\alpha_{n-1}+1})] \\
&\quad \boxplus [\delta(p_{\alpha_{n-1}+1}, \epsilon, p_{\alpha_{n-1}+2}) \boxplus \cdots \boxplus \delta(p_{\alpha_n-1}, \epsilon, p_{\alpha_n}) \boxplus T(p_{\alpha_n})]
\end{aligned}$$

where $k_i \leq |Q|$. So $|M^e|_d = |M|_d$ by Proposition 2.3. $\square$

Corollary 4.3. *$L_d(\mathcal{E}, \Sigma) = \{|M|_d \in \mathcal{E}^{\Sigma^*} : M \in \epsilon NFA(\mathcal{E}, \Sigma)\}$.*

5. Closure properties of $\mathcal{E}$-valued languages

To compare with the closure properties of $\mathcal{E}$-valued languages in the depth-first mode, we re-examine the closure properties of $\mathcal{E}$-valued languages in the width-first mode in this section.

Theorem 5.1 (*Intersection Operation [37]*)**.** *$L_d(\mathcal{E}, \Sigma)$ is closed under the intersection operation.*

Theorem 5.2 (*Intersection Operation*)**.** *$L_w(\mathcal{E}, \Sigma)$ is closed under the intersection operation.*

Proof. Suppose that $M_1 = (Q_1, \Sigma, I_1, T_1, \delta_1)$ and $M_2 = (Q_2, \Sigma, I_2, T_2, \delta_2)$ are two $\mathcal{E}$-valued automata with $Q_1 \cap Q_2 = \phi$. The languages they recognize are L_1 and L_2, respectively.

Construct an $\mathcal{E}$-valued automaton $M_1 \wedge M_2 = (Q_1 \cup Q_2, \Sigma, I^{M_1 \wedge M_2}, T^{M_1 \wedge M_2}, \delta^{M_1 \wedge M_2})$ as follows:

$$I^{M_1 \wedge M_2}(p) = \begin{cases} I_1(p), & \text{if } p \in Q_1 \\ I_2(p), & \text{if } p \in Q_2 \end{cases}$$

$$T^{M_1 \wedge M_2}(p) = \begin{cases} T_1(p), & \text{if } p \in Q_1 \\ T_2(p), & \text{if } p \in Q_2 \end{cases}$$

$$\delta^{M_1 \wedge M_2}(p, \sigma, q) = \begin{cases} \delta_1(p, \sigma, q), & \text{if } p, q \in Q_1 \\ \delta_2(p, \sigma, q), & \text{if } p, q \in Q_2 \\ 1, & \text{otherwise} \end{cases}$$

By Eq. (6), for $s=\sigma_1\cdots\sigma_n\in\Sigma^+$,

$$\begin{aligned}|M_1\wedge M_2|_w(s) &= \wedge_{p_n}\left(\cdots\left(\wedge_{p_0}I^{M_1\wedge M_2}(p_0)\boxplus\delta^{M_1\wedge M_2}(p_0,\sigma_1,p_1)\right)\cdots\right)\boxplus T^{M_1\wedge M_2}(p_n)\\ &= \wedge_{p_n\in Q_1}\left(\cdots\left(\wedge_{p_0}I^{M_1\wedge M_2}(p_0)\boxplus\delta^{M_1\wedge M_2}(p_0,\sigma_1,p_1)\right)\cdots\right)\boxplus T_1(p_n)\\ &\quad\bigwedge\wedge_{p_n\in Q_2}\left(\cdots\left(\wedge_{p_0}I^{M_1\wedge M_2}(p_0)\boxplus\delta^{M_1\wedge M_2}(p_0,\sigma_1,p_1)\right)\cdots\right)\boxplus T_2(p_n)\\ &= \wedge_{p_n\in Q_1}\left(\cdots\left(\wedge_{p_0\in Q_1}I_1(p_0)\boxplus\delta_1(p_0,\sigma_1,p_1)\right)\cdots\right)\boxplus T_1(p_n)\\ &\quad\bigwedge\wedge_{p_n\in Q_2}\left(\cdots\left(\wedge_{p_0\in Q_2}I_2(p_0)\boxplus\delta_2(p_0,\sigma_1,p_1)\right)\cdots\right)\boxplus T_2(p_n)\\ &= |M_1|_w(s)\bigwedge|M_2|_w(s)\end{aligned}$$

and $|M_1\wedge M_2|_w(\epsilon)=|M_1\wedge M_2|_d(\epsilon)=|M_1|_d(\epsilon)\wedge|M_2|_d(\epsilon)=|M_1|_w(\epsilon)\wedge|M_2|_w(\epsilon)$ by Theorem 5.1. □

Suppose $M_1=(Q_1,\Sigma,I_1,T_1,\delta_1)$ and $M_2=(Q_2,\Sigma,I_2,T_2,\delta_2)$ are $\mathcal{E}$-valued automata with $Q_1\cap Q_2=\phi$. Construct an $\mathcal{E}$NFA $M_1\boxplus M_2=(Q_1\times Q_2,\Sigma,I^{M_1\boxplus M_2},T^{M_1\boxplus M_2},\delta^{M_1\boxplus M_2})$, where

$$I^{M_1\boxplus M_2}(p,q)=I_1(p)\boxplus I_2(q)$$
$$T^{M_1\boxplus M_2}(p,q)=T_1(p)\boxplus T_2(q)$$
$$\delta^{M_1\boxplus M_2}((p_0,q_0),\sigma,(p_1,q_1))=\delta_1(p_0,\sigma,p_1)\boxplus\delta_2(q_0,\sigma,q_1)$$

Theorem 5.3 (*Disjoint Sum Operation [37]*)**.** *Let $M_1=(Q_1,\Sigma,I_1,T_1,\delta_1)$ and $M_2=(Q_2,\Sigma,I_2,T_2,\delta_2)$ be two $\mathcal{E}$NFA. If one of the Q_1,Q_2 contains at least two states, then $|M_1\boxplus M_2|_d=|M_1|_d\boxplus|M_2|_d$ iff $(a\boxplus b)\wedge(a\boxplus c)=a\boxplus(b\wedge c)$ for any $a,b,c\in\mathcal{E}$.*

Theorem 5.4 (*Disjoint Sum Operation*)**.** *For any two $\mathcal{E}$NFA $M_1=(Q_1,\Sigma,I_1,T_1,\delta_1)$ and $M_2=(Q_2,\Sigma,I_2,T_2,\delta_2)$, if one of the Q_1,Q_2 contains at least two states, then $|M_1\boxplus M_2|_w=|M_1|_w\boxplus|M_2|_w$ iff $(a\boxplus b)\wedge(a\boxplus c)=a\boxplus(b\wedge c)$ for any $a,b,c\in\mathcal{E}$.*

Proof. "If part". If $\boxplus$ distributes over $\wedge$, then $|M_1\boxplus M_2|_w=|M_1\boxplus M_2|_d=|M_1|_d\boxplus|M_2|_d=|M_1|_w\boxplus|M_2|_w$ by Proposition 2.4 and Theorem 5.3.

"Only if part". Consider the two $\mathcal{E}$NFA $M_1=(\{p_0,p_1\},\Sigma,I_1,T_1,\delta_1)$ and $M_2=(\{p_3,p_4\},\Sigma,I_2,T_2,\delta_2)$. Suppose $\sigma\in\Sigma$, define

$$\begin{array}{ll} I_1(p_0)=a,\ I_1(p_1)=\mathbf{0} & I_2(p_3)=I_2(p_4)=\mathbf{0}\\ T_1(p_0)=T_1(p_1)=\mathbf{0} & T_2(p_3)=T_2(p_4)=\mathbf{0}\\ \delta_1(p_0,\sigma,p_0)=\delta_1(p_0,\sigma,p_1)=\mathbf{0} & \delta_2(p_3,\sigma,p_4)=b,\ \delta_2(p_4,\sigma,p_3)=c\end{array}$$

All other values of δ_1 and δ_2 are $\mathbf{1}$. Then $|M_1|_w(\sigma)=a$ and $|M_2|_w(\sigma)=b\wedge c$. It is easy to see that $|M_1\boxplus M_2|_w(\sigma)=\wedge_{(q_1,r_1)}\left(\wedge_{(q_0,r_0)}I^{M_1\boxplus M_2}(q_0,r_0)\boxplus\delta^{M_1\boxplus M_2}((q_0,r_0),\sigma,(q_1,r_1))\right)\boxplus T^{M_1\boxplus M_2}(q_1,r_1)=(a\boxplus b)\wedge(a\boxplus c)$. From $(|M_1|_w\boxplus|M_2|_w)(\sigma)=|M_1\boxplus M_2|_w(\sigma)$, then $a\boxplus(b\wedge c)=(a\boxplus b)\wedge(a\boxplus c)$. □

Corollary 5.5. *If $\mathcal{E}$ is an MV algebra, then $L_w(\mathcal{E},\Sigma)$ is closed under the sum operation.*

Theorem 5.6 (*Reversal Operation [37]*)**.** *$L_d(\mathcal{E},\Sigma)$ is closed under the reversal operation.*

Suppose $M=(Q,\Sigma,I,T,\delta)$ is an $\mathcal{E}$NFA. Construct an $\mathcal{E}$NFA $M^R=(Q,\Sigma,I^R,T^R,\delta^R)$ as follows: $I^R(p)=T(p)$, $T^R(p)=I(p)$ and $\delta^R(p,\sigma,q)=\delta(q,\sigma,p)$.

Theorem 5.7 (*Reversal Operation*)**.** *$L_w(\mathcal{E},\Sigma)$ is closed under reversal if $(a\boxplus b)\wedge(a\boxplus c)=a\boxplus(b\wedge c)$ for any $a,b,c\in\mathcal{E}$. Conversely, $(a\boxplus b)\wedge(a\boxplus c)=a\boxplus(b\wedge c)$ for any $a,b,c\in\mathcal{E}$ if $|M|_w=|M^R|_w$ for any $\mathcal{E}$NFA.*

Proof. If $\boxplus$ distributes over $\wedge$, then $L_d(\mathcal{E},\Sigma)=L_w(\mathcal{E},\Sigma)$ by Proposition 2.4. Thus $L_w(\mathcal{E},\Sigma)$ is closed by Theorem 5.6.

If $|M|_w=|M^R|_w$ for any $\mathcal{E}$NFA, assume $\sigma\in\Sigma$ and consider $M=(\{p_0,p_1,p_2,p_3\},\Sigma,I,T,\delta)$ as follows:

$$I(p_0)=I(p_1)=\mathbf{0},\ I(p_2)=I(p_3)=\mathbf{1}$$
$$T(p_0)=T(p_1)=T(p_2)=\mathbf{1},\ T(p_3)=\mathbf{0}$$
$$\delta(p_0,\sigma,p_2)=b,\ \delta(p_1,\sigma,p_2)=c,\ \delta(p_2,\sigma,p_3)=a$$

All other values of δ are $\mathbf{1}$. We have that $|M|_w(\sigma\sigma)=(\delta(p_0,\sigma,p_2)\wedge(p_1,\sigma,p_2))\boxplus\delta(p_2,\sigma,p_3)=(b\wedge c)\boxplus a$ and $|M^R|_w(\sigma\sigma)=(\delta^R(p_3,\sigma,p_2)\boxplus\delta^R(p_2,\sigma,p_0))\wedge(\delta^R(p_3,\sigma,p_2)\boxplus\delta^R(p_2,\sigma,p_1))=(a\boxplus b)\wedge(a\boxplus c)$. Thus $(a\boxplus b)\wedge(a\boxplus c)=a\boxplus(b\wedge c)$ for all $a,b,c\in\mathcal{E}$. □

Suppose $M_1=(Q_1,\Sigma,I_1,T_1,\delta_1)$ and $M_2=(Q_2,\Sigma,I_2,T_2,\delta_2)$ are $\mathcal{E}$NFAs with $Q_1\cap Q_2=\phi$. Construct an $\mathcal{E}\epsilon$NFA $M_1\cdot M_2=(Q,\Sigma,I^{M_1\cdot M_2},T^{M_1\cdot M_2},\delta^{M_1\cdot M_2})$ as $Q=Q_1\cup Q_2$,

$$I^{M_1\cdot M_2}(p)=\begin{cases}I_1(p), & \text{if } p\in Q_1\\ \mathbf{1}, & \text{if } p\in Q_2,\end{cases}\qquad T^{M_1\cdot M_2}(p)=\begin{cases}\mathbf{1}, & \text{if } p\in Q_1\\ T_2(p), & \text{if } p\in Q_2\end{cases}$$

$$\delta^{M_1\cdot M_2}(p,\sigma,q)=\begin{cases}\delta_i(p,\sigma,q), & \text{if } p,q\in Q_i\\ T_1(p)\boxplus I_2(q), & \text{if } \sigma=\epsilon,\ (p,q)\in Q_1\times Q_2\\ \mathbf{1}, & \text{otherwise}\end{cases}$$

Theorem 5.8 (*Concatenation Operation*)**.** *Let M_1 and M_2 be $\mathcal{E}$NFA. The followings are equivalent:*

(i) $(a \boxplus b) \wedge (a \boxplus c) = a \boxplus (b \wedge c)$ *for all* $a, b, c \in \mathcal{E}$.
(ii) $|M_1 \cdot M_2|_d = |M_1|_d \cdot |M_2|_d$ *for any* M_1 *and* M_2.
(iii) $|M_1 \cdot M_2|_w = |M_1|_w \cdot |M_2|_w$ *for any* M_1 *and* M_2.

Proof. (i)⇒(ii): Suppose $M_1 = (Q_1, \Sigma, I_1, T_1, \delta_1)$ and $M_2 = (Q_2, \Sigma, I_2, T_2, \delta_2)$ and $Q_1 \cap Q_2 = \phi$. For any $s = \sigma_1 \cdots \sigma_n \in \Sigma^+$,

$$\begin{aligned} |M_1 \cdot M_2|_d(s) &= \wedge_{0\le k\le n} \wedge_{p_i} I^{M_1\cdot M_2}(p_0) \boxplus \delta^{M_1\cdot M_2}(p_0, \sigma_1, p_1) \boxplus \cdots \boxplus \delta^{M_1\cdot M_2}(p_k, \epsilon, p_{k+1}) \\ &\quad \boxplus \cdots \boxplus \delta^{M_1\cdot M_2}(p_n, \sigma_n, p_{n+1}) \boxplus T^{M_1\cdot M_2}(p_{n+1}) \\ &= \wedge_{0\le k\le n}[(\wedge_{p_i} I_1(p_0) \boxplus \cdots \boxplus T_1(p_k)) \boxplus (\wedge_{p_j} I_2(p_{k+1}) \boxplus \cdots \boxplus T_2(p_{n+1}))] \\ &= \wedge_{s_1 s_2 = s}[|M_1|(s_1) \boxplus |M_2|(s_2)] \\ &= (|M_1|_d \cdot |M_2|_d)(s) \end{aligned}$$

for $s = \epsilon$,

$$\begin{aligned} |M_1 \cdot M_2|_d(\epsilon) &= \wedge_{p,q} I^{M_1\cdot M_2}(p) \boxplus \delta^{M_1\cdot M_2}(p, \epsilon, q) \boxplus T^{M_1\cdot M_2}(q) \\ &= \wedge_{p\in Q_1, q\in Q_2} I_1(p) \boxplus \delta^{M_1\cdot M_2}(p, \epsilon, q) \boxplus T_2(q) \\ &= \wedge_{p\in Q_1, q\in Q_2} I_1(p) \boxplus T_1(p) \boxplus I_2(q) \boxplus T_2(q) \\ &= (\wedge_{p\in Q_1} I_1(p) \boxplus T_1(p)) \boxplus (\wedge_{q\in Q_2} I_2(q) \boxplus T_2(q)) \\ &= |M_1|(\epsilon) \boxplus |M_2|(\epsilon) \\ &= (|M_1|_d \cdot |M_2|_d)(\epsilon) \end{aligned}$$

(ii)⇒(i): For any $a, b, c \in \mathcal{E}$, consider $M_1 = (\{p\}, \Sigma, I_1, T_1, \delta_1)$, $M_2 = (\{q_1, q_2\}, \Sigma, I_2, T_2, \delta_2)$ as following, for some $\sigma \in \Sigma$:

$$I_1(p) = a, T_1(p) = \mathbf{0}, \delta_1(p, \sigma, p) = \mathbf{1}$$
$$I_2 = T_2 = \mathbf{0}, \delta_2(q_1, \sigma, q_1) = b, \delta_2(q_1, \sigma, q_2) = c, \delta_2(q_2, \sigma, \cdot) = \mathbf{1}$$

then $(|M_1|_d \cdot |M_2|_d)(\sigma) = (|M_1|_d(\epsilon) \boxplus |M_2|_d(\sigma)) \wedge (|M_1|_d(\sigma) \boxplus |M_2|_d(\epsilon)) = a \boxplus (b \wedge c)$.

$$\begin{aligned} |M_1 \cdot M_2|_d(\sigma) &= \wedge_{p_i}\{[I^{M_1\cdot M_2}(p_0) \boxplus \delta^{M_1\cdot M_2}(p_0, \epsilon, p_1) \boxplus \delta^{M_1\cdot M_2}(p_1, \sigma, p_2) \boxplus T^{M_1\cdot M_2}(p_2)] \\ &\quad \wedge [I^{M_1\cdot M_2}(p_0) \boxplus \delta^{M_1\cdot M_2}(p_0, \sigma, p_1) \boxplus \delta^{M_1\cdot M_2}(p_1, \epsilon, p_2) \boxplus T^{M_1\cdot M_2}(p_2)]\} \\ &= (I_1(p) \boxplus T_1(p) \boxplus I_2(q_1) \boxplus \delta_2(q_1, \sigma, q_1) \boxplus T_2(q_1)) \wedge (I_1(p) \boxplus T_1(p) \boxplus I_2(q_1) \boxplus \delta_2(q_1, \sigma, q_2) \boxplus T_2(q_2)) \\ &= (a \boxplus b) \wedge (a \boxplus c) \end{aligned}$$

Thus $(a \boxplus b) \wedge (a \boxplus c) = a \boxplus (b \wedge c)$ by the hypothesis.

(i)⇒(iii): From the above proof, we know (ii) is true when (i) holds. By Proposition 2.4, then $|M_1 \cdot M_2|_w = |M_1 \cdot M_2|_d = |M_1|_d \cdot |M_2|_d = |M_1|_w \cdot |M_2|_w$.

(iii)⇒(i): Consider $M_1 = (\{p_0, p_1\}, \Sigma, I_1, T_1, \delta_1)$ and $M_2 = (\{q_0, q_1, q_2\}, \Sigma, I_2, T_2, \delta_2)$ as following:

$$\begin{array}{ll} I_1(p_0) = \mathbf{0}, I_1(p_1) = \mathbf{1} & I_2(q_0) = \mathbf{0}, I_2(q_1) = I_2(q_2) = \mathbf{1} \\ T_1(p_0) = \mathbf{1}, T_1(p_1) = \mathbf{0} & T_2(q_0) = \mathbf{1}, T_2(q_1) = T_2(q_2) = \mathbf{0} \\ \delta_1(p_0, \sigma, p_1) = a & \delta_2(q_0, \sigma, q_1) = b, \delta_2(q_0, \sigma, q_2) = c \end{array}$$

δ_1 and δ_2 take value $\mathbf{1}$ for other arguments. We get that $|M_1|_w(\sigma) = a$, $|M_1|_w(\epsilon) = \mathbf{1}$ and $|M_2|_w(\sigma) = b \wedge c$, $|M_2|_w(\epsilon) = \mathbf{1}$, so $(|M_1|_w \cdot |M_2|_w)(\sigma\sigma) = a \boxplus (b \wedge c)$. On the other hand, since $\delta_w^{M_1\cdot M_2}(I^{M_1\cdot M_2}, \epsilon) = \delta_w^{M_1\cdot M_2}(I^{M_1\cdot M_2}, \sigma\sigma) = \delta_w^{M_1\cdot M_2}(I^{M_1\cdot M_2}, \sigma\epsilon\epsilon) = \mathbf{1}$, there is

$$\delta_w^{M_1\cdot M_2}(I^{M_1\cdot M_2}, \epsilon t) = \delta_w^{M_1\cdot M_2}(I^{M_1\cdot M_2}, \sigma\sigma t) = \delta_w^{M_1\cdot M_2}(I^{M_1\cdot M_2}, \sigma\epsilon\epsilon t) = \mathbf{1}$$

for any $t \in \mathcal{H}_\Sigma$. And

$$\delta_w^{M_1\cdot M_2}(I^{M_1\cdot M_2}, \sigma\epsilon\sigma)(r) = \begin{cases} a \boxplus b, & \text{if } r = q_1 \\ a \boxplus c, & \text{if } r = q_2 \\ \mathbf{1}, & \text{otherwise} \end{cases}$$

implies $\delta_w^{M_1\cdot M_2}(I^{M_1\cdot M_2}, \sigma\epsilon\sigma\epsilon^k) = \mathbf{1}$. So we only need take into account $t = \sigma\epsilon\sigma$ in the following calculation:

$$\begin{aligned} |M_1 \cdot M_2|_w(\sigma\sigma) &= \wedge_{||t||=\sigma\sigma} \wedge_r \delta^{M_1\cdot M_2}(I^{M_1\cdot M_2}, t)(r) \boxplus T^{M_1\cdot M_2}(r) \\ &= \wedge_r \delta^{M_1\cdot M_2}(I^{M_1\cdot M_2}, \sigma\epsilon\sigma)(r) \boxplus T^{M_1\cdot M_2}(r) \\ &= (a \boxplus b) \wedge (a \boxplus c) \end{aligned}$$

So $|M_1 \cdot M_2|_w(\sigma\sigma) = (|M_1|_w \cdot |M_2|_w)(\sigma\sigma)$ leads to $(a \boxplus b) \wedge (a \boxplus c) = a \boxplus (b \wedge c)$. □

Let Σ and Δ be two alphabets. A mapping $f : \Sigma^* \longrightarrow \Delta^*$ is called a homomorphism if $f(uv) = f(u)f(v)$ for all $u, v \in \Sigma^*$. Naturally $f(\epsilon) = \epsilon$ if f is a homomorphism. The homomorphism f is determined by the image of Σ since Σ^* is the free monoid generated by Σ. Furthermore, the homomorphism f introduces a mapping $f^* : \mathcal{E}^{\Sigma^*} \longrightarrow \mathcal{E}^{\Delta^*}$ as follows: $f^*(\varphi)(t) = \wedge\{\varphi(s) : f(s) = t\}$ for any $\varphi \in \mathcal{E}^{\Sigma^*}$ and $t \in \Delta^*$.

Theorem 5.9 (*Homomorphic Image*). *Let $f : \Sigma^* \longrightarrow \Delta^*$ be a homomorphism. $f^*(|M|_d) \in L_d(\mathcal{E}, \Delta)$ for any $M \in$NFA$(\mathcal{E}, \Sigma)$.*

Proof. At first we introduce some denotations: assume $M = (Q, \Sigma, I, T, \delta)$, where $Q = \{p_1, \ldots, p_l\}$, $\Sigma = \{\sigma_1, \ldots, \sigma_k\}$. Denote $f(\sigma_i) = t_i = \omega_{i,1} \cdots \omega_{i,n_i} \in \Delta^+$ in which case $n_i \geq 1$, $i = 1, \ldots, k$. Let $n_i = 0$ if $t_i = \epsilon$. Define a class of disjoint sets $Q_i^{a,b} = \{q_{i,0}^{a,b}, \ldots, q_{i,n_i}^{a,b}\}$ for $a, b = 1, \ldots, l$, $i = 1, \ldots, k$. Denote $Q' = \bigcup_{a,b,i} Q_i^{a,b}$.

Construct $\widetilde{M} = (\widetilde{Q}, \Delta, \widetilde{I}, \widetilde{T}, \widetilde{\delta}) \in \epsilon$NFA$(\mathcal{E}, \Delta)$ as follows: $\widetilde{Q} = Q \cup Q'$,

$$\widetilde{I}(r) = \begin{cases} I(r), & \text{if } r \in Q \\ \mathbf{1}, & \text{otherwise,} \end{cases} \qquad \widetilde{T}(r) = \begin{cases} T(r), & \text{if } r \in Q \\ \mathbf{1}, & \text{otherwise} \end{cases}$$

$$\widetilde{\delta} : \begin{cases} \widetilde{\delta}(p_a, \epsilon, q_{i,0}^{a,b}) = \delta(p_a, \sigma_i, p_b) \\ \widetilde{\delta}(q_{i,j-1}^{a,b}, \omega_{i,j}, q_{i,j}^{a,b}) = \mathbf{0} \quad \text{for } j = 1, \ldots, n_i \text{ if } n_i \geq 1 \\ \widetilde{\delta}(q_{i,n_i}^{a,b}, \epsilon, p_b) = \mathbf{0} \\ \widetilde{\delta} = \mathbf{1} \quad \text{for other cases} \end{cases}$$

Next we show $f^*(|M|_d) = |\widetilde{M}|_d$. Suppose $t = \omega_1 \cdots \omega_n \in \Delta^*$, $t' = \omega_1' \cdots \omega_m' \in \mathcal{H}_\Delta$. Since $|\widetilde{M}|_d(t) = \wedge_{||t'||=t} \wedge_{r_i \in \widetilde{Q}} \widetilde{I}(r_0) \boxplus \widetilde{\delta}(r_0, \omega_1', r_1) \boxplus \cdots \boxplus \widetilde{\delta}(r_{m-1}, \omega_m', r_m) \boxplus \widetilde{T}(r_m)$, we take a sight into the calculation of each

$$\widetilde{I}(r_0) \boxplus \widetilde{\delta}(r_0, \omega_1', r_1) \boxplus \cdots \boxplus \widetilde{\delta}(r_{m-1}, \omega_m', r_m) \boxplus \widetilde{T}(r_m) \tag{8}$$

First we can assume $r_0, r_m \in Q$ by the definition of $\widetilde{I}$ and $\widetilde{T}$, otherwise Eq. (8) $= \mathbf{1}$. Suppose $r_{x_0} = r_0, r_{x_1}, \ldots, r_{x_h} = r_m (x_i < x_{i+1}, h \geq 1)$ are all the states in $\{r_1, \ldots, r_m\}$ belonging to Q. It is necessary that $\widetilde{\delta}(r_0, \omega_1', r_1) \neq \mathbf{1}$ requires $\omega_1' = \epsilon$, $r_0 = p_a$, $r_1 = q_{i,0}^{a,b} \in Q'$ for some a, b, i, so $\widetilde{\delta}(r_0, \omega_1', r_1) = \delta(p_a, \sigma_i, p_b)$.

If $n_i \geq 1$, $\widetilde{\delta}(r_j, \omega_{j+1}', r_{j+1}) \neq \mathbf{1}$ requires $\omega_{j+1}' = \omega_{i,j}$, $r_{j+1} = q_{i,j}^{a,b}$ for $j = 1, \ldots, n_i$, so $\widetilde{\delta}(r_j, \omega_{j+1}', r_{j+1}) = \widetilde{\delta}(q_{i,j-1}^{a,b}, \omega_{i,j}, q_{i,j}^{a,b}) = \mathbf{0}$. $\widetilde{\delta}(r_{n_i+1}, \omega_{n_i+2}', r_{n_i+2}) \neq \mathbf{1}$ requires $\omega_{n_i+2}' = \epsilon$, $r_{n_i+2} = p_b$, so $\widetilde{\delta}(r_{n_i+1}, \omega_{n_i+2}', r_{n_i+2}) = \widetilde{\delta}(q_{i,n_i}^{a,b}, \epsilon, p_b) = \mathbf{0}$.

If $n_i = 0$, $\widetilde{\delta}(r_1, \omega_2', r_2) \neq \mathbf{1}$ requires $\omega_2' = \epsilon$, $r_2 = p_b$, so $\widetilde{\delta}(r_1, \omega_2', r_2) = \widetilde{\delta}(q_{i,0}^{a,b}, \epsilon, p_b) = \mathbf{0}$. Here $f(\sigma_i) = \epsilon = \omega_2'$.

Therefore $\widetilde{\delta}(r_{x_0}, \omega_1', r_1) \boxplus \cdots \boxplus \widetilde{\delta}(r_{x_1-1}, \omega_{x_1}', r_{x_1}) = \delta(p_a, \sigma_i, p_b)$, $r_{x_0} = p_a$, $r_{x_1} = p_b \in Q$, $\omega_1' = \epsilon$, $f(\sigma_i) = \omega_{i,1} \cdots \omega_{i,n_i} = \omega_2' \cdots \omega_{n_i+1}' \in \Delta^*$.

With the same discussion above, $\widetilde{\delta}(r_{x_{u-1}}, \omega_{x_{u-1}+1}', r_{x_{u-1}+1}) \boxplus \cdots \boxplus \widetilde{\delta}(r_{x_u-1}, \omega_{x_u}', r_{x_u}) = \delta(p_{a_u}, \sigma_{i_u}, p_{b_u})$ and $r_{x_{u-1}} = p_{a_u}$, $r_{x_u} = p_{b_u}$ for some a_u, b_u, i_u, $u = 1, \ldots, h$. Note that $b_u = a_{u+1}$. Eq. (8) $\neq \mathbf{1}$ implies

$$\text{Eq. (8)} = I(p_{a_1}) \boxplus \delta(p_{a_1}, \sigma_{i_1}, p_{b_1}) \boxplus \cdots \boxplus \delta(p_{a_h}, \sigma_{i_h}, p_{b_h}) \boxplus T(p_{b_h}) \tag{9}$$

We get $t' = \epsilon f(\sigma_{i_1}) \cdots \epsilon f(\sigma_{i_h})$ by $\omega_{x_{u-1}+1}' = \epsilon$ and $f(\sigma_{i_u}) = \omega_{x_{u-1}+2}' \cdots \omega_{x_u}'$. Denote $s = \sigma_{i_1} \cdots \sigma_{i_h}$, $f(s) = f(\sigma_{i_1}) \cdots f(\sigma_{i_h}) = ||t'|| = t$. Eq. (9) is a $\wedge$-item of $|M|_d(s) = \wedge_{r_i} I(r_0) \boxplus \delta(r_0, \sigma_{i_1}, r_1) \boxplus \cdots \boxplus \delta(r_{h-1}, \sigma_{i_h}, r_h) \boxplus T(r_h)$. So Eq. (8)$\geq \wedge\{|M|_d(s) : f(s) = t\}$, it follows that $|\widetilde{M}|_d(t) \geq f^*(|M|_d)(t)$.

On the other side, assume $f(s) = t$, $s = \sigma_{i_1} \cdots \sigma_{i_h}$ and $(p_{a_1}, p_{a_2} = p_{b_1}, \ldots, p_{b_h})$ is an h-path of M. Define a set of indices

$$\begin{cases} x_0 = 0, \\ x_u = x_{u-1} + ||f(\sigma_{i_u})|| + 1, \quad u = 1, \ldots, h \end{cases}$$

Denote $\omega_{x_{u-1}+1}' = \epsilon$ and $f(\sigma_{i_u}) = \omega_{x_{u-1}+2}' \cdots \omega_{x_u}' \in \Delta^+$ for each σ_{i_u} $(u = 1, \ldots, h)$. Let $m = \sum_{u=1}^{h}(||f(\sigma_{i_u})|| + 1)$. Select an m-path $(r_0, \ldots, r_m)$ of $t' = \epsilon f(\sigma_{i_1}) \cdots \epsilon f(\sigma_{i_h})$:

$$\begin{cases} r_{x_u} = p_{b_u} = p_{a_{u+1}}, \quad u = 0, \ldots, h \\ r_{x_u+j} = p_{i_u,j-1}^{a_{u+1},b_{u+1}}, \quad j = 1, \ldots, f(\sigma_{i_u}) \end{cases}$$

Thus

$$\widetilde{I}(r_0) \boxplus \widetilde{\delta}(r_0, \omega_{x_0+1}', r_1) \boxplus \cdots \boxplus \widetilde{\delta}(r_{m-1}, \omega_{x_h}', r_m) \boxplus \widetilde{T}(r_m) = I(p_{a_1}) \boxplus \delta(p_{a_1}, \sigma_{i_1}, p_{b_1}) \boxplus \cdots \boxplus \delta(p_{a_h}, \sigma_{i_h}, p_{b_h}) \boxplus T(p_{b_h})$$

by the definition of $\widetilde{\delta}$. Note that $f(s) = t = ||t'||$, so $f^*(|M|_d)(t) = \wedge\{|M|_d(s) : f(s) = t\} \geq |\widetilde{M}|_d(t)$. Therefore $|\widetilde{M}|_d(t) = f^*(|M|_d)(t)$. □

Definition 5.1. The complement of an $\mathcal{E}$-valued language L is $L^\perp(s) = (L(s))'$ where $'$ is the complement operation in $\mathcal{E}$.

Theorem 5.10 (*Complement*). *$L_d(\mathcal{E}, \Sigma)$ and $L_w(\mathcal{E}, \Sigma)$ are not closed under complementation.*

Proof. Let $\mathcal{E}$ be the special linear ordered MV algebra defined in [5]. It consists of a set of formal symbols: $\{0, c, c+c, \dots, 1-c-c, 1-c, 1\}$, $0 \cdot c = 0$ and $n \cdot c = c + c + \dots + c$ n-times and $1 - 0 \cdot c = 1$ and $1 - n \cdot c = 1 - c - c - \dots c$ n-times. The rules of the operations were described in [5]. Since c is not idempotent if $c \neq 0$, there is $n \cdot c \neq m \cdot c$ when $n \neq m$. In the following, we find that just non-idempotence of c leads to the contradiction. Define an $\mathcal{E}$DFA $M = (\{p_0, p_1\}, \{\sigma\}, I, T, \delta)$ as $I(p_0) = 0, I(p_1) = 1, T(p_0) = 1, T(p_1) = 0, \delta(p_0, \sigma, p_1) = \delta(p_1, \sigma, p_1) = c$, and $\delta = 1$ for the rest. It is easy to calculate that $|M|_d(\sigma^n) = n \cdot c$. If there exists some $\mathcal{E}$NFA $M^{\perp} = (Q^{\perp}, \{\sigma\}, I^{\perp}, T^{\perp}, \delta^{\perp})$ satisfying $|M^{\perp}|_d = (|M|_d)'$, then $|M^{\perp}|_d(\sigma^n) = 1 - n \cdot c$. Since $\mathcal{E}$ is linearly ordered, $|M^{\perp}|_d(\sigma^n) = \wedge I^{\perp}(q_0) \boxplus \delta^{\perp}(q_0, \sigma, q_1) \boxplus \dots \boxplus T^{\perp}(q_n) = I^{\perp}(r_0) \boxplus \delta^{\perp}(r_0, \sigma, r_1) \boxplus \dots \boxplus T^{\perp}(r_n)$. When $n \geq |Q| + 1$, there exist $i < j$ such that $r_i = r_j$. Hence $|M^{\perp}|_d(\sigma^n) > I^{\perp}(r_0) \boxplus \dots \boxplus \delta^{\perp}(r_{i-1}, \sigma, r_i) \boxplus \delta^{\perp}(r_j, \sigma, r_{j+1}) \boxplus \dots \boxplus T^{\perp}(r_n) \geq |M^{\perp}|_d(\sigma^m)$ where $m = n - (j - i) < n$. That leads to $|M|_d(\sigma^n) < |M|_d(\sigma^m)$, which contradicts $|M|_d(\sigma^n) = n \cdot c > m \cdot c = |M|_d(\sigma^m)$. Since M is an $\mathcal{E}$DFA, then $|M|_w(\sigma^n) = |M|_d(\sigma^n) = n \cdot c$. If there exists some $\mathcal{E}$NFA $M^* = (Q^*, \{\sigma\}, I^*, T^*, \delta^*)$ satisfying $|M^*|_w = (|M|_w)'$, then $|M^*|_w(\sigma^n) = 1 - n \cdot c$. Since $\mathcal{E}$ is linearly ordered, there exists $\tilde{p}_0$ such that $\wedge_{p_0} I^*(p_0) \boxplus \delta^*(p_0, \sigma, p_1) = I^*(\tilde{p}_0) \boxplus \delta^*(\tilde{p}_0, \sigma, p_1)$ for any given p_1. Continuing this way,

$$
\begin{aligned}
|M^*|_w(\sigma^n) &= \bigwedge_{p_n} \left(\cdots \left(\bigwedge_{p_0} I^*(p_0) \boxplus \delta^*(p_0, \sigma, p_1) \right) \cdots \right) \boxplus T^*(p_n) \\
&= \bigwedge_{p_n} \left(\cdots \left(I^*(\tilde{p}_0) \boxplus \delta^*(\tilde{p}_0, \sigma, p_1) \right) \cdots \right) \boxplus T^*(p_n) \\
&\ \ \vdots \\
&= I^*(\tilde{p}_0) \boxplus \delta^*(\tilde{p}_0, \sigma, \tilde{p}_1) \boxplus \dots \boxplus T^*(\tilde{p}_n)
\end{aligned}
$$

When $n \geq |Q| + 1$, there exist $i < j$ such that $r_i = r_j$. Hence similar to the depth-first mode, we can obtain the same contradiction. □

If $\mathcal{E}$ degenerates into an orthomodular lattice, then the operation $\boxplus$ degenerates into being the disjunction operation $\vee$. In the following, we will see that the complementation will have a different representation in sharp quantum automata.

Lemma 5.11. *Let $\mathcal{E}$ be an orthomodular lattice. For any $M \in NFA(\mathcal{E}, \Sigma)$, there is a $M' \in NFA(\mathcal{E}, \Sigma)$ with classical initial states and classical transitions such that $|M'|_d = |M|_d$.*

Proof. Denote $R_M = \{r_1 \vee \dots \vee r_n : r_i \in R(I) \cup R(T) \cup R(\delta)\}$, where R means the range of function. Obviously R_M is finite. Construct $M' = (Q \times R, \Sigma, I', T', \delta')$ as: $I'((p, I(p))) = 0$ and $I' = 1$ for others; $\delta'((p_1, e_1), \sigma, (p_2, e_2)) = 0$ if $\delta(p_1, \sigma, p_2) < 1$ and $e_2 = e_1 \vee \delta(p_1, \sigma, p_2)$; finally $T'((p, e)) = e \vee T(p)$. The initial state function and transitions of M' are classical. It is straightforward to check that $|M'|_d = |M|_d$. □

Lemma 5.12. *Let $\mathcal{E}$ be an orthomodular lattice, $M \in NFA(\mathcal{E}, \Sigma)$ and M' be the $\mathcal{E}$NFA constructed in Lemma 5.11. $|M|_w = |M'|_w$ for $\forall M \in NFA(\mathcal{E}, \Sigma)$ if and only if $\vee$ distributes over $\wedge$.*

Proof. If $\vee$ distributes over $\wedge$, then $|M|_w = |M|_d$ for all $M \in$ NFA$(\mathcal{E}, \Sigma)$. So $|M|_w = |M'|_w$ by Lemma 5.11.

On the other hand, for any $a, b, c \in \mathcal{E}$, let $M = (\{q_1, q_2, q_3, \}, \Sigma, I, T, \delta)$ in which,

$$
\begin{aligned}
&I(q_1) = I(q_2) = 0, \qquad I(q_3) = 1 \\
&T(q_1) = T(q_2) = 1, \qquad T(q_3) = c \\
&\delta(q_1, \sigma, q_3) = a, \qquad \delta(q_2, \sigma, q_3) = b
\end{aligned}
$$

and $\delta = 1$ for the rest. Then $|M|_w(\sigma) = [(I(q_1) \vee \delta(q_1, \sigma, q_3)) \wedge (I(q_2) \vee \delta(q_2, \sigma, q_3))] \vee T(q_3) = (a \wedge b) \vee c$.

By the construction,

$$
\begin{aligned}
|M'|_w(\sigma) &= \bigwedge_{(p_1, e_1)} \left(\bigwedge_{p_0} I'((p_0, I(p_0))) \vee \delta((p_0, I(p_0)), \sigma, (p_1, e_1)) \right) \vee T'((p_1, e_1)) \\
&= \big(I'((q_1, 0)) \vee \delta((q_1, 0), \sigma, (q_3, a)) \vee T'((q_3, a))\big) \bigwedge \big(I'((q_2, 0)) \vee \delta((q_2, 0), \sigma, (q_3, b)) \vee T'((q_3, b))\big) \\
&= T'((q_3, a)) \bigwedge T'((q_3, b)) \\
&= (a \vee c) \wedge (b \vee c) = |M|_w(\sigma) = (a \wedge b) \vee c \quad \square
\end{aligned}
$$

Proposition 5.13. *Let $\mathcal{E}$ be an orthomodular lattice. For any $M \in DFA(\mathcal{E}, \Sigma)$ with single initial state there exists an $M^{\perp} \in DFA(\mathcal{E}, \Sigma)$ with single initial state such that $|M^{\perp}|_d = (|M|_d)'$ and $|M^{\perp}|_w = (|M|_w)'$.*

Proof. Let $M = (Q, \Sigma, I, T, \delta)$. By Lemma 5.11 we could assume there exists $q_I \in Q$ such that $I(q_I) = 0$ and $I(q) = 1$ for $q \neq q_I$. Also there is only one q satisfying $\delta(p, a, q) = 0$ for each pair (p, q), and $\delta(p, a, q) = 1$ for other q. Then $|M|_d(s) = T(p)$ where p is the last state of the path along which M accepts s. Construct $M^{\perp} = (Q, \Sigma, I, T^{\perp}, \delta), T^{\perp}(p) = (T(p))'$. Note that for any given s, the paths in M and in $M^{\perp}$ are the same. Thus $|M|_d(s) = T(p) = (T^{\perp}(p))' = (|M^{\perp}|_d(s))'$. Since there is only one path for any input s, so $|M|_d = |M|_w$ and $|M^{\perp}|_d = |M^{\perp}|_w$, that is $|M^{\perp}|_w = (|M|_w)'$. □

Certainly, when $\mathcal{E}$ is an orthomodular lattice and depth-first mode is used, the closure of the complementation is treated in [25].

From Theorem 5.10 to Proposition 5.13, we found that merely the deficiency of idempotence for a QMV algebra caused that the complementation operation of $\mathcal{E}$-valued language does not exist. From the algebraic point of view, this is obviously a main difference between automata theory based on sharp quantum logic and automata theory based on unsharp quantum logic.

6. $\mathcal{E}$-valued regular expressions

The Kleene theorem, which gives the equivalence between finite-state automata and regular expressions, is one of the most important results in classical automata theory. The proof of the Kleene theorem provides a way to transform a finite state automaton into a regular expression. In this section, we present that transformation in the framework of $\mathcal{E}$NFA and compare the power of accepting language between $\mathcal{E}$NFA and their $\mathcal{E}$-valued regular expressions. We find out that if they have the same power of accepting languages, then the truth valued lattice should degenerate into an MV algebra.

In the following, we assume that $\mathcal{E}$ is a complete lattice in this section.

Definition 6.1. An $\mathcal{E}$-valued regular expression u over Σ is an element of $(\Sigma \cup \mathcal{E} \cup \{\epsilon, \phi\} \cup \{+, \cdot, \infty, (,)\})^*$, the $\mathcal{E}$-valued language of u is denoted by $|u|$. $\mathcal{E}$-valued regular expressions are defined inductively as following:

(i) $u = \phi$ is an $\mathcal{E}$-valued regular expression over Σ; $|\phi|(s) = \mathbf{1}$ for all $s \in \Sigma^*$.
(ii) $u = \epsilon$ is an $\mathcal{E}$-valued regular expression over Σ; $|\epsilon|(s) = \mathbf{0}$ if $s = \epsilon$ and $|\epsilon|(s) = \mathbf{1}$ otherwise.
(iii) $u = \sigma \in \Sigma$ is an $\mathcal{E}$-valued regular expression over Σ; $|\sigma|(s) = \mathbf{0}$ if $s = \sigma$ and $|\sigma|(s) = \mathbf{1}$ otherwise.
(iv) If v is an $\mathcal{E}$-valued regular expression over Σ, then for any $\lambda, \mu \in \mathcal{E}$, $u_1 = (\lambda \cdot v)$, $u_2 = (v \cdot \mu)$ are $\mathcal{E}$-valued regular expressions over Σ, their languages are $|u_1|(s) = \lambda \boxplus |v|(s)$, $|u_2|(s) = |v|(s) \boxplus \mu$ respectively.
(v) If v, w are $\mathcal{E}$-valued regular expressions over Σ, then
  (a) $u = (v + w)$ is an $\mathcal{E}$-valued regular expression over Σ; $|u|(s) = |v|(s) \wedge |w|(s)$ for all $s \in \Sigma^*$;
  (b) $u = (v \cdot w)$ is an $\mathcal{E}$-valued regular expression over Σ; $|u| = |v| \cdot |w|$, i.e. the concatenation of $|v|$ and $|w|$; Denote $u^1 = u$, $u^{k+1} = (u^k \cdot u)$ for $k \geq 1$;
  (c) $u = (v^\infty)$ is an $\mathcal{E}$-valued regular expression over Σ; $|u| = |v| \wedge |v^2| \wedge \cdots$.

Note that $(u \cdot v) \cdot w \neq u \cdot (v \cdot w)$ in general.

Similarly to the automata theory based on sharp quantum logic [40,41], the only difference between $\mathcal{E}$-valued regular expressions and the classical ones is that there are additional unary (scalar) operators $\lambda \in \mathcal{E}$.

Definition 6.2. For any $\mathcal{E}$NFA $M = (Q, \Sigma, I, T, \delta)$, where $Q = \{p_1, \ldots, p_n\}$, define a series $\mathcal{E}$-valued regular expressions $R_{i,j}^k$ $(k = 0, \ldots, n; i, j = 1, \ldots, n)$ over Σ as follows:

- when $k = 0$, define $R_{i,j}^0 = \begin{cases} \epsilon + \sum_\sigma \delta(p_i, \sigma, p_j) \cdot \sigma, & \text{if } i = j \\ \sum_\sigma \delta(p_i, \sigma, p_j) \cdot \sigma, & \text{otherwise} \end{cases}$
- when $1 \leq k \leq n$, define $R_{i,j}^k = R_{i,j}^{k-1} + R_{i,k}^{k-1} \cdot R_{k,j}^{k-1} + [R_{i,k}^{k-1} \cdot (R_{k,k}^{k-1})^+] \cdot R_{k,j}^{k-1}$

Proposition 6.1. *Assume that u is a $\mathcal{E}$-valued regular expression over Σ, if $\boxplus$ distributes over $\wedge$ then $|u^\infty|(s) = |u|(s) \wedge |u^2|(s) \wedge \cdots \wedge |u^n|(s)$ for all $s \in \Sigma^n$.*

Proof. Let $s = \sigma_1 \cdots \sigma_n$,

$$|u^k|(s) = \bigwedge_{||s_1 \cdots s_k|| = s, s_i \in \Sigma^*} |u|(s_1) \boxplus \cdots \boxplus |u|(s_k)$$

for all $k \geq n + 1$. Since the length of s is n, there must exist empty sequences in $s_1, \ldots, s_k$. Denote these nonempty sequences by $s_{i_1}, \ldots, s_{i_l}$, $(l \leq n)$ in their original order. Since $|u|(s_1) \boxplus \cdots \boxplus |u|(s_k) \geq |u|(s_{i_1}) \boxplus \cdots \boxplus |u|(s_{i_l})$, obviously each $|u|(s_{i_1}) \boxplus \cdots \boxplus |u|(s_{i_l})$ is contained in the expanded form of $|u^l|(s)$. So

$$|u^k|(s) \geq |u|(s) \wedge |u^2|(s) \wedge \cdots \wedge |u^n|(s)$$

and $|u^\infty|(s) = |u|(s) \wedge |u^2|(s) \wedge \cdots \wedge |u^n|(s)$. $\square$

Theorem 6.2. *Let $M = (Q, \Sigma, I, T, \delta) \in$NFA$(\mathcal{E}, \Sigma)$. Then the followings are equivalent:*

(i) *$a \boxplus (b \wedge c) = (a \boxplus b) \wedge (a \boxplus c)$ for any $a, b, c \in \mathcal{E}$.*
(ii) *$|M|_d = |\sum_{i,j} I(p_i) \cdot R_{i,j}^n \cdot T(p_j)|$ for any $M \in NFA(\mathcal{E}, \Sigma)$.*
(iii) *$|M|_w = |\sum_{i,j} I(p_i) \cdot R_{i,j}^n \cdot T(p_j)|$ for any $M \in NFA(\mathcal{E}, \Sigma)$.*

Proof. (i)⇒(ii): By the definition above,

$$|R^0_{i,j}|(s) = \begin{cases} \mathbf{1}, & \text{if } s = \epsilon,\ i \neq j \\ \mathbf{0}, & \text{if } s = \epsilon,\ i = j \\ \delta(p_i, s, p_j), & \text{if } s \in \Sigma \\ \mathbf{1}, & \text{if } |s| \geq 2 \end{cases}$$

(1) If $|s| = l \leq 1$, first we prove $|R^n_{i,j}|(s) = |R^0_{i,j}|(s)$ by induction. Obviously $|R^h_{i,j}|(s) = |R^0_{i,j}|(s)$ if $h = 0$. Assume $|R^h_{i,j}|(s) = |R^0_{i,j}|(s)$ for $h < n$ by hypothesis. When $h = n$,

$$\begin{aligned} |R^h_{i,j}|(\epsilon) &= |R^{n-1}_{i,j}|(\epsilon) \wedge |R^{n-1}_{i,n} \cdot R^{n-1}_{n,j}|(\epsilon) \wedge |R^{n-1}_{i,n} \cdot (R^{n-1}_{n,n})^+ \cdot R^{n-1}_{n,j}|(\epsilon) \\ &= |R^0_{i,j}|(\epsilon) \wedge |R^0_{i,n} \cdot R^0_{n,j}|(\epsilon) \wedge |R^0_{i,n} \cdot (R^0_{n,n})^+ \cdot R^0_{n,j}|(\epsilon) \\ &= |R^0_{i,j}|(\epsilon) \end{aligned}$$

For any $\sigma \in \Sigma$,

$$\begin{aligned} |R^h_{i,j}|(\sigma) &= |R^{n-1}_{i,j}|(\sigma) \wedge |R^{n-1}_{i,n} \cdot R^{n-1}_{n,j}|(\sigma) \wedge |R^{n-1}_{i,n} \cdot (R^{n-1}_{n,n})^+ \cdot R^{n-1}_{n,j}|(\sigma) \\ &= |R^0_{i,j}|(\sigma) \wedge |R^0_{i,n} \cdot R^0_{n,j}|(\sigma) \wedge |R^0_{i,n} \cdot (R^0_{n,n})^+ \cdot R^0_{n,j}|(\sigma) \\ &= \delta(p_i, \sigma, p_j) \\ &= |R^0_{i,j}|(\sigma) \end{aligned}$$

Furthermore, $|\sum_{i,j} I(p_i) \cdot R^n_{i,j} \cdot T(p_j)|(\epsilon) = \bigwedge_{i,j} I(p_i) \boxplus |R^0_{i,j}|(\epsilon) \boxplus T(p_j) = \bigwedge_i I(p_i) \boxplus T(p_i) = |M|(\epsilon)$. In a similar way, $|\sum_{i,j} I(p_i) \cdot R^n_{i,j} \cdot T(p_j)|(\sigma) = \bigwedge_{i,j} I(p_i) \boxplus |R^0_{i,j}|(\sigma) \boxplus T(p_j) = \bigwedge_{i,j} I(p_i) \boxplus \delta(p_i, \sigma, p_j) \boxplus T(p_i) = |M|(\sigma)$.

(2) If $|s| = l \geq 2$, denote $s = \sigma_1 \cdots \sigma_l$. We show that

$$|R^n_{i,j}|(s) = \bigwedge_{q_1,\ldots,q_{l-1} \in Q} \delta(p_i, \sigma_1, q_1) \boxplus \cdots \boxplus \delta(q_{l-1}, \sigma_l, p_j)$$

When $h = 1$, if $l = 2$

$$\begin{aligned} |R^h_{i,j}|(s) &= |R^0_{i,j}|(s) \wedge |R^0_{i,1} \cdot R^0_{1,j}|(s) \wedge |R^0_{i,1} \cdot (R^1_{1,1})^+ \cdot R^0_{1,j}|(s) \\ &= |R^0_{i,1} \cdot R^0_{1,j}|(s) \wedge |R^0_{i,1} \cdot (R^0_{1,1})^+ \cdot R^0_{1,j}|(s) \\ &= \delta(p_i, \sigma, q_1) \boxplus \delta(q_1, \sigma, p_j) \\ &= \bigwedge_{q_1,\ldots,q_{l-1} \in \{p_1\}} \delta(p_i, \sigma_1, q_1) \boxplus \cdots \boxplus \delta(q_{l-1}, \sigma_l, p_j) \end{aligned}$$

if $l \geq 3$

$$\begin{aligned} |R^h_{i,j}|(s) &= |R^0_{i,j}|(s) \wedge |R^0_{i,1} \cdot R^0_{1,j}|(s) \wedge |R^0_{i,1} \cdot (R^1_{1,1})^+ \cdot R^0_{1,j}|(s) \\ &= |R^0_{i,1} \cdot R^0_{1,j}|(s) \wedge |R^0_{i,1} \cdot (R^0_{1,1})^+ \cdot R^0_{1,j}|(s) \\ &= |R^0_{i,1}|(\sigma_1) \boxplus |(R^0_{1,1})^+|(\sigma_2 \cdots \sigma_{l-1}) \boxplus |R^0_{1,j}|(\sigma_l) \\ &= |R^0_{i,1}|(\sigma_1) \boxplus |R^0_{1,1}|(\sigma_2) \boxplus \cdots \boxplus |R^0_{1,1}|(\sigma_{l-1}) \boxplus |R^0_{1,j}|(\sigma_l) \\ &= \delta(p_i, \sigma_1, p_1) \boxplus \delta(p_1, \sigma_2, p_1) \boxplus \cdots \boxplus \delta(p_1, \sigma_{l-1}, p_1) \boxplus \delta(p_1, \sigma_l, p_j) \\ &= \bigwedge_{q_1,\ldots,q_{l-1} \in \{p_1\}} \delta(p_i, \sigma_1, q_1) \boxplus \cdots \boxplus \delta(q_{l-1}, \sigma_l, p_j) \end{aligned}$$

The hypothesis holds for $h < n$. In the case of $h = n$,

$$\begin{aligned} |R^h_{i,j}|(s) &= |R^{n-1}_{i,j}|(s) \wedge |R^{n-1}_{i,n} \cdot R^{n-1}_{n,j}|(s) \wedge |R^{n-1}_{i,n} \cdot (R^{n-1}_{n,n})^+ \cdot R^{n-1}_{n,j}|(s) \\ &= |R^{n-1}_{i,j}|(s) \wedge |R^{n-1}_{i,n} \cdot R^{n-1}_{n,j}|(s) \wedge |R^{n-1}_{i,n} \cdot (R^{n-1}_{n,n} + (R^{n-1}_{n,n})^2 + \cdots + (R^{n-1}_{n,n})^{l-2}) \cdot R^{n-1}_{n,j}|(s) \\ &= |R^{n-1}_{i,j}|(s) \wedge |R^{n-1}_{i,n} \cdot R^{n-1}_{n,j}|(s) \wedge |R^{n-1}_{i,n} \cdot R^{n-1}_{n,n} \cdot R^{n-1}_{n,j}|(s) \wedge \cdots \wedge |R^{n-1}_{i,n} \cdot (R^{n-1}_{n,n})^{l-2} \cdot R^{n-1}_{n,j}|(s) \\ &= \wedge\{\delta(p_i, \sigma_1, q_1) \boxplus \cdots \boxplus \delta(q_{l-1}, \sigma_l, p_j) | q_1, \ldots, q_{l-1} \in Q, \text{no } q_i \text{ is } p_n\} \\ &\quad \wedge \{\delta(p_i, \sigma_1, q_1) \boxplus \cdots \boxplus \delta(q_{l-1}, \sigma_l, p_j) | q_1, \ldots, q_{l-1} \in Q, \text{one } q_i \text{ is } p_n\} \cdots \\ &\quad \wedge \{\delta(p_i, \sigma_1, q_1) \boxplus \cdots \boxplus \delta(q_{l-1}, \sigma_l, p_j) | q_1, \ldots, q_{l-1} \in Q, \text{every } q_i \text{ is } p_n\} \\ &= \bigwedge_{q_1,\ldots,q_{l-1} \in Q} \delta(p_i, \sigma_1, q_1) \boxplus \cdots \boxplus \delta(q_{l-1}, \sigma_l, p_j) \end{aligned}$$

So we have proved $|\sum_{i,j} I(p_i)\cdot R^n_{i,j}\cdot T(p_j)|(s) = \bigwedge_{q_0,\dots,q_l\in Q} I(q_0)\boxplus\delta(q_0,\sigma_1,q_1)\boxplus\cdots\boxplus\delta(q_{l-1},\sigma_l,q_l)\boxplus T(q_l) = |M|(s)$. Hence $f = |\sum_{i,j} I(p_i)\cdot R^n_{i,j}\cdot T(p_j)|$.

(ii)⇒(i): For any $a, b, c \in \mathcal{E}$, consider $M = (\{p_1, p_2, p_3\}, \Sigma, I, T, \delta)$ where $\sigma \in \Sigma$,

$$I(p_1) = a, I(p_2) = I(p_3) = \mathbf{1}$$
$$T(p_1) = T(p_2) = \mathbf{1}, T(p_3) = \mathbf{0}$$
$$\delta(p_1,\sigma,p_2) = b, \delta(p_1,\sigma,p_3) = c, \delta(p_2,\sigma,p_3) = \delta(p_3,\sigma,p_3) = \mathbf{0}$$

and $\delta = \mathbf{1}$ for other cases. It is easy to see that $|\sum_{i,j} I(p_i)\cdot R^3_{i,j}\cdot T(p_j)|(\sigma\sigma) = |I(p_1)\cdot R^3_{1,3}\cdot T(p_3)|(\sigma\sigma) = a\boxplus(b\wedge c)$. On the other hand, $|M|_d(\sigma\sigma) = (a\boxplus b)\wedge(a\boxplus c)$. Then it is concluded that $\boxplus$ distributes over $\wedge$ from $|M|_d = |\sum_{i,j} I(p_i)\cdot R^n_{i,j}\cdot T(p_j)|$.

(i)⇒(iii): From the above proof, we know (ii) is true when (i) holds. By Proposition 2.4, then $|M|_w = |M|_d = |\sum_{i,j} I(p_i)\cdot R^n_{i,j}\cdot T(p_j)|$.

(iii)⇒(i): For any $a, b, c \in \mathcal{E}$, consider the M constructed in (ii)⇒(i),

$$\begin{aligned} |M|_w(\sigma\sigma) &= \wedge_{q_2}(\wedge_{q_1}(\wedge_{q_0} I(q_0)\boxplus\delta(q_0,\sigma,q_1))\boxplus\delta(q_1,\sigma,q_2))\boxplus T(q_2) \\ &= [(I(p_1)\boxplus\delta(p_1,\sigma,p_2)\boxplus\delta(p_2,\sigma,p_3))\wedge(I(p_1)\boxplus\delta(p_1,\sigma,p_3)\boxplus\delta(p_3,\sigma,p_3))]\boxplus T(p_3) \\ &= (a\boxplus b)\wedge(a\boxplus c) \end{aligned}$$

Therefore, $\boxplus$ distributes over $\wedge$ if $|M|_w = |\sum_{i,j} I(p_i)\cdot R^n_{i,j}\cdot T(p_j)|$ for any $M \in$NFA$(\mathcal{E}, \Sigma)$. □

7. $\mathcal{E}$-valued context free grammars

Recall that a classical context-free grammar (CFG) is $G = (V, \Sigma, S, P)$ where

(i) Σ is a finite set of terminal symbols.
(ii) V is a finite set of nonterminal symbols and $V \cap \Sigma = \phi$.
(iii) $S \in V$ is the start symbol.
(iv) P is a finite subset of $PROD(G) = \{A \to \gamma : A \in V, \gamma \in (\Sigma \cup V)^*\}$, whose elements are called productions.

For a context-free grammar $G = (V, P, S)$, the direct derivation relation "⇒" is defined to a binary relation over $\Gamma := (\Sigma \cup V)^*$ as: $\alpha \Rightarrow \beta$ iff $\alpha = \alpha_1 A\alpha_2, \beta = \alpha_1\gamma\alpha_2$ for some $A \to \gamma \in P, \alpha_i \in \Gamma$. Meanwhile $\alpha \Rightarrow \beta$ and $A \to \gamma$ are said to be compatible with each other, denoted by $(\alpha \Rightarrow \beta) \approx (A \to \gamma)$ or $(A \to \gamma) \approx (\alpha \Rightarrow \beta)$. Let $\overset{*}{\Rightarrow}$ denote the reflexive and transitive closure of ⇒. More specifically, if $\alpha_1, \alpha_2, \dots, \alpha_m \in \Gamma$, and $\alpha_i \Rightarrow \alpha_{i+1}, i = 1, 2, \dots, m-1$, then we say $\alpha_1 \overset{*}{\Rightarrow} \alpha_m$ or α_1 derives α_m in grammar G. Then the language $L(G)$ generated by G is defined by

$$L(G) = \{\varpi \in \Sigma^* : S \overset{*}{\Rightarrow} \varpi\} \tag{10}$$

Similarly, we can define the notion of $\mathcal{E}$-valued context free grammar.

Definition 7.1. Let $\mathcal{E}$ be a lattice ordered QMV algebra. An $\mathcal{E}$-valued context free grammar ($\mathcal{E}$CFG) is a quadruple $G = (V, \Sigma, S, \mu)$, where V and S are the same with classical context free grammars, μ is a map: $PROD(G) \longrightarrow \mathcal{E}$ where its support $supp(\mu) = \{p \in PROD(G) : \mu(p) \neq \mathbf{1}\}$ is finite.

Compared with the classical definition, there are two principles of defining the languages generated by an $\mathcal{E}$-valued context free grammar — a depth-first mode and a width-first mode. The difference between them is in how we evaluate the truth value of the proposition that a word is generated from the truth values of the propositions related to the involved transitions. In classical automata theory, distributivity of boolean logic warrants that these two ways are equivalent. However, they are not equivalent in the case of $\mathcal{E}$-valued context free grammar unless the truth valued lattice is an MV algebra.

First, we consider the depth-first mode. If $\alpha = \gamma_1 A\gamma_2, \beta = \gamma_1\gamma\gamma_2$ and there is $p = A \to \gamma \in PROD(G)$, it is said $\alpha \Rightarrow \beta$ is compatible with p or p is compatible with $\alpha \Rightarrow \beta$, denoted as $(\alpha \Rightarrow \beta) \approx p$ or $p \approx (\alpha \Rightarrow \beta)$. If $p = A \to \gamma \in PROD(G)$, we denote $b(p) = A, e(p) = \gamma$. The class of all $\mathcal{E}$CFG over Σ is denoted by CFG$(\mathcal{E}, \Sigma)$.

Definition 7.2 (*n-derivation*)**.** Let $\alpha, \beta \in \Gamma$. If $n \geq 1$, an n-derivation between α and β is an element $\kappa = ((\alpha_0 = \alpha, \alpha_1, \dots, \alpha_n = \beta), (p_1, p_2, \dots p_n)) \in \Gamma^{n+1} \times PROD(G)^n$ where $(\alpha_{i-1} \Rightarrow \alpha_i) \approx p_i$. The $\mathcal{E}$-value of κ is $|\kappa| = \sum_i \mu(p_i)$. The set of all n-derivations between α and β in G is denoted by $D^n_G(\alpha, \beta)$. Denote $D^0_G(\alpha, \beta) = \phi$ and $D^+_G(\alpha, \beta) = \bigcup_n D^n_G(\alpha, \beta)$.

Remark 7.1. Order of the production may be exchanged. Let

$$\kappa = ((\alpha_0, \alpha_1, \dots, \alpha_n), (p_1, \dots, p_n)) \in D^+_G(S, s)$$

Suppose that there is $\alpha_k = \beta_1 A_1\beta_2 A_2\beta_3, \alpha_{k+1} = \beta_1 A_1\beta_2\gamma_2\beta_3, \alpha_{k+2} = \beta_1\gamma_1\beta_2\gamma_2\beta_3$ where $\beta_i \in \Gamma, p_{k+1} = A_2 \to \gamma_2, p_{k+2} = A_1 \to \gamma_1$. Obviously, the production p_{k+1} is used firstly, then p_{k+2}. Consider the n-derivation

$$\kappa' = ((\alpha_0, \dots, \alpha_k, \alpha'_{k+1}, \alpha_{k+2}, \dots, \alpha_n), (p_1, \dots, p_{k+2}, p_{k+1}, \dots, p_n))$$

where $\alpha'_{k+1} = \beta_1\gamma_1\beta_2A_2\beta_2$. The production p_{k+2} is used firstly, then p_{k+1}. It is easy to see that $|\kappa| = |\kappa'|$. So the order of production does not affect the value of κ.

Definition 7.3. For any $G \in \mathrm{CFG}(\mathcal{E}, \Sigma)$, the $\mathcal{E}$-valued language generated by G in the depth-first mode is defined to be an $\mathcal{E}$-valued map $|G|_d$, and it is given by

$$|G|_d(s) = \bigwedge\{|\kappa| : \kappa \in D^+_G(S, s)\} \quad \text{for } \forall s \in \Sigma^+ \tag{11}$$

Now, we define the language generated by an $\mathcal{E}$-valued CFG in the width-first mode.

Let $\overset{0}{\Rightarrow}$ be the identity relation over Γ. The composition of $\Rightarrow$ is defined as: $\overset{n+1}{\Rightarrow} := \overset{n}{\Rightarrow} \circ \Rightarrow$. Define $\overset{*}{\Rightarrow} := \bigcup_{n\geq 0} \overset{n}{\Rightarrow}$. A word w could be generated by G iff $S \overset{n}{\Rightarrow} w$ for some $n \in \mathbf{Z}^+$.

Definition 7.4. Let $G \in \mathrm{CFG}(\mathcal{E}, \Sigma)$. The $\mathcal{E}$-value of the derivation "$\Rightarrow$" is characterized by $\widetilde{\mu} : \overset{*}{\Rightarrow} \longrightarrow \mathcal{E}$:

$$\begin{aligned}
\widetilde{\mu}(\alpha \overset{0}{\Rightarrow} \beta) &= \begin{cases} \mathbf{0}, & \text{if } \alpha = \beta \\ \mathbf{1}, & \text{otherwise} \end{cases} \\
\widetilde{\mu}(\alpha \Rightarrow \beta) &= \bigwedge\{\mu(p) : (\alpha \Rightarrow \beta) \approx p, p \in supp(\mu)\} \\
\widetilde{\mu}(\alpha \overset{n+1}{\Rightarrow} \beta) &= \bigwedge_{\gamma\in\Gamma} \widetilde{\mu}(\alpha \overset{n}{\Rightarrow} \gamma) \boxplus \widetilde{\mu}(\gamma \Rightarrow \beta) \quad \text{for } n \geq 1 \\
\widetilde{\mu}(\alpha \overset{*}{\Rightarrow} \beta) &= \bigwedge_{m\geq 1} \widetilde{\mu}(\alpha \overset{m}{\Rightarrow} \beta)
\end{aligned}$$

We make an agreement that $\widetilde{\mu}(\alpha \Rightarrow \beta) = \mathbf{1}$ if $\{p \in supp(\mu) : (\alpha \Rightarrow \beta) \approx p\} = \phi$.

Definition 7.5. Let $G \in \mathrm{CFG}(\mathcal{E}, \Sigma)$. The $\mathcal{E}$-valued language $|G|_w$ generated by G in the width-first mode is defined as:

$$|G|_w(s) = \bigwedge_{m\geq 1} \widetilde{\mu}(S \overset{m}{\Rightarrow} s) \quad \text{for } \forall s \in \Sigma^* \tag{12}$$

Eq. (12) could be written in a more computable way:

$$|G|_w(s) = \bigwedge_{m\geq 1} \left(\bigwedge_{\gamma_{m-1}\in\Gamma} \left(\cdots \left(\bigwedge_{\gamma_1\in\Gamma} \widetilde{\mu}(S \Rightarrow \gamma_1) \boxplus \widetilde{\mu}(\gamma_1 \Rightarrow \gamma_2) \right) \cdots \right) \boxplus \widetilde{\mu}(\gamma_{m-1} \Rightarrow s) \right) \tag{13}$$

Remark 7.2. In the following we demand $\mathcal{E}$ to be a complete lattice in order to make Eqs. (11) and (12) well defined.

Theorem 7.1. *$|G|_d = |G|_w$ for $\forall G \in \mathrm{CFG}(\mathcal{E}, \Sigma)$ iff $a \boxplus (b \wedge c) = (a \boxplus b) \wedge (a \boxplus c)$ for any $a, b, c \in \mathcal{E}$.*

Proof. If $\boxplus$ distributes over $\wedge$, denote $\gamma_0 = S$, $\gamma_m = s$,

$$\begin{aligned}
|G|_w(s) &= \bigwedge_{m\geq 1} \bigwedge_{\gamma_i\in\Gamma} \widetilde{\mu}(S \Rightarrow \gamma_1) \boxplus \cdots \boxplus \widetilde{\mu}(\gamma_{m-1} \Rightarrow s) \\
&= \bigwedge_{m\geq 1} \bigwedge_{\gamma_i\in\Gamma} (\wedge\{\mu(p_1) : p_1 \approx (S \Rightarrow \gamma_1)\}) \boxplus \cdots \boxplus (\wedge\{\mu(p_m) : p_m \approx (\gamma_{m-1} \Rightarrow s)\}) \\
&= \bigwedge_{m\geq 1} \bigwedge_{\gamma_i\in\Gamma} \bigwedge\{\mu(p_1) \boxplus \cdots \boxplus \mu(p_m) : p_i \approx (\gamma_{i-1} \Rightarrow \gamma_i), i = 1, \ldots, m\} \\
&= \bigwedge_{m\geq 1} \bigwedge_{\gamma_i\in\Gamma} \bigwedge\{|\kappa| : \kappa = ((S, \gamma_1, \ldots, \gamma_m = s), (p_1, \ldots, p_m))\} \\
&= \bigwedge_{m\geq 1} \bigwedge\{|\kappa| : \kappa \in D^m_G(S, s)\} \\
&= |G|_d(s)
\end{aligned}$$

Conversely, if $|G|_d = |G|_w$ for $\forall G \in \mathrm{CFG}(\mathcal{E}, \Sigma)$, consider $G = (\{S, A, B, C\}, \Sigma, S, \mu)$ where

$$\begin{aligned}
&\mu(S \to \sigma A) = a, \mu(A \to \sigma C) = \mathbf{0} \\
&\mu(S \to \sigma B) = b, \mu(B \to \sigma C) = \mathbf{0} \\
&\mu(C \to \sigma) = c
\end{aligned}$$

and $\mu = \mathbf{1}$ for other cases. Then $|G|_d(\sigma\sigma\sigma) = (a \boxplus c) \wedge (b \boxplus c) = |G|_w = [(\widetilde{\mu}(S \Rightarrow \sigma A) \boxplus \widetilde{\mu}(\sigma A \Rightarrow \sigma\sigma C)) \wedge (\widetilde{\mu}(S \Rightarrow \sigma B) \boxplus \widetilde{\mu}(\sigma B \Rightarrow \sigma\sigma C))] \boxplus \widetilde{\mu}(\sigma\sigma C \Rightarrow \sigma\sigma\sigma) = (a \wedge b) \boxplus c$. □

In the classical automata theory, Chomsky normal form (CNF) and Greibach normal form (GNF) are two useful special forms of context-free grammars. In the following, we consider their $\mathcal{E}$-valued generalization.

Definition 7.6. Let $G = (V, \Sigma, S, \mu) \in \mathrm{CFG}(\mathcal{E}, \Sigma)$. Denote $PROD_C(G) = \{p : p = A \to BC, A, B, C \in V\} \cup \{p : p = A \to \sigma, A \in V, \sigma \in \Sigma_\epsilon\}$. Then G is said to be in $\mathcal{E}$-valued Chomsky normal form ($\mathcal{E}$CNF) if $supp(\mu) \subseteq PROD_C(G)$.

Definition 7.7. Let $G = (V, \Sigma, S, \mu) \in \mathrm{CFG}(\mathcal{E}, \Sigma)$. Denote $PROD_G(G) = \{p : p = A \to \sigma\alpha, \sigma \in \Sigma, \alpha \in V^*\} \cup \{p : p = A \to \epsilon, A \in V\}$. Then G is said to be in $\mathcal{E}$-valued Greibach normal form ($\mathcal{E}$GNF) if $supp(\mu) \subseteq PROD_G(G)$.

For any $G = (V, \Sigma, S, \mu) \in \mathrm{CFG}(\mathcal{E}, \Sigma)$, construct a $G^C = (V^C, \Sigma, S, \mu^C)$ which is in $\mathcal{E}$CNF recursively as follows: for any $A \to \gamma \in supp(\mu) - PROD_C(G)$, if

(1) Denote $G_0 = (V_0, \Sigma, S, \mu_0) = (V, \Sigma, S, \mu) = G$.
(2) Denote $\Phi_i = \{p = A \to \sigma\alpha : \sigma \in \Sigma, \alpha \in (\Sigma \cup V_i)^+\} \cap supp(\mu_i)$, $\Psi_i = \{p = A \to B\alpha : \alpha \in (\Sigma \cup V_i)^+ - V\} \cap supp(\mu_i)$ for any $G_i = (V_i, \Sigma, S, \mu_i)$.
(3) If $\Phi_i \neq \phi$, suppose $p = A \to \sigma\alpha \in \Phi_i$, introduce two new symbols $B, C \notin V_i$. Define $G_{i+1} = (V_{i+1}, \Sigma, S, \mu_{i+1})$ where $V_{i+1} = V_i \cup \{B, C\}$, μ_{i+1} is defined as

$$\mu_{i+1}(A \to BC) = \mu_i(A \to \sigma\alpha),\ \mu_{i+1}(B \to \sigma) = \mu_{i+1}(C \to \alpha) = \mathbf{0},\ \mu_{i+1}(A \to \sigma\alpha) = \mathbf{1}$$

and $\mu_{i+1}(p) = \mu_i(p)$ for any $p \in supp(\mu_i) - \{A \to \sigma\alpha\}$, $\mu_{i+1}(p) = \mathbf{1}$ for any $p \in PROD(G_{i+1}) - supp(\mu_i) - \{A \to BC, B \to \sigma, C \to \alpha\}$.

Since $supp(G_i)$ is finite for each G_i, it is easy to see that there is $\Phi_{N_0} = \phi$ for some positive integer N_0. Then execute the following recursive process starting from G_{N_0}:

(4) If $\Psi_j \neq \phi$, suppose $p = A \to B\alpha \in \Psi_j$, introduce a new symbol $D \notin V_j$. Define $G_{j+1} = (V_{j+1}, \Sigma, S, \mu_{j+1})$ where $V_{j+1} = V_j \cup \{D\}$, μ_{j+1} is defined as

$$\mu_{j+1}(A \to BD) = \mu_j(A \to B\alpha),\ \mu_{j+1}(D \to \alpha) = \mathbf{0},\ \mu_{j+1}(A \to B\alpha) = \mathbf{1}$$

and $\mu_{j+1}(p) = \mu_j(p)$ for any $p \in supp(\mu_j) - \{A \to B\alpha\}$, $\mu_{j+1}(p) = \mathbf{1}$ for any $p \in PROD(G_{j+1}) - supp(\mu_j) - \{A \to BD, D \to \alpha\}$.

Similarly, we get $\Psi_{N_1} = \phi$ for some positive $N_1 \geq N_0$ since $supp(G_j)$ is finite. We define $G^C = G_{N_1}$.

The following theorem establishes the generalization of Chomsky normal form in unsharp quantum logic. It is easy to see that the distributive law is not required in the depth-first mode but is needed in the width-first mode.

Theorem 7.2. (i) $|G|_d = |G^C|_d$ *for any* $G \in CFG(\mathcal{E}, \Sigma)$.
(ii) $|G|_w = |G^C|_w$ *for any* $G \in CFG(\mathcal{E}, \Sigma)$ *iff* $a \boxplus (b \wedge c) = (a \boxplus b) \wedge (a \boxplus c)$ *for any* $a, b, c \in \mathcal{E}$.

Proof. (i) It is straightforward to verify that $|G|_d = |G^C|_d$.

(ii) Let $G = (\{S, A, B\}, \Sigma, S, \mu)$ and $a, b, c \in \mathcal{E}$. Assume $\sigma \in \Sigma$, define $\mu(S \to AB) = a$, $\mu(A \to A\sigma) = b$, $\mu(B \to \sigma B) = c$, $\mu(A \to \sigma) = \mu(B \to \sigma) = \mathbf{0}$ and $supp(\mu) = \{S \to AB, A \to A\sigma, A \to \sigma, B \to \sigma B, B \to \sigma\}$. Consider the $\mathcal{E}$-value of $s = \sigma\sigma\sigma$:

$$\begin{aligned}
|G|_w(s) &= \wedge_{m\geq 1}\left(\wedge_{\gamma_{m-1}\in\Gamma}\left(\cdots\left(\wedge_{\gamma_1\in\Gamma}\widetilde{\mu}(S \Rightarrow \gamma_1) \boxplus \widetilde{\mu}(\gamma_1 \Rightarrow \gamma_2)\right)\cdots\right) \boxplus \widetilde{\mu}(\gamma_{m-1} \Rightarrow s)\right)\\
&= \wedge_{m\geq 1}(\wedge_{\gamma_{m-1}\in\Gamma}\left(\cdots\left(\wedge_{\gamma_2\in\Gamma}\mu(S \to AB) \boxplus \widetilde{\mu}(AB \Rightarrow \gamma_2) \boxplus \widetilde{\mu}(\gamma_2 \Rightarrow \gamma_3)\right)\cdots\right) \boxplus \widetilde{\mu}(\gamma_{m-1} \Rightarrow s))\\
&= \wedge_{m\geq 1}(\wedge_{\gamma_{m-1}\in\Gamma}(\cdots([\mu(S \to AB) \boxplus (\mu(A \to A\sigma) \wedge \mu(B \to \sigma B)) \boxplus \widetilde{\mu}(A\sigma B \Rightarrow \gamma_3)]\\
&\quad \wedge [\mu(S \to AB) \boxplus \mu(A \to \sigma) \boxplus \widetilde{\mu}(\sigma B \Rightarrow \gamma_3)]\\
&\quad \wedge [\mu(S \to AB) \boxplus \mu(B \to \sigma) \boxplus \widetilde{\mu}(A\sigma \Rightarrow \gamma_3)])\cdots) \boxplus \widetilde{\mu}(\gamma_{m-1} \Rightarrow s))\\
&= \{([\mu(S \to AB) \boxplus (\mu(A \to A\sigma) \wedge \mu(B \to \sigma B) \boxplus \mu(A \to \sigma)]\\
&\quad \wedge [\mu(S \to AB) \boxplus \mu(A \to \sigma) \boxplus \mu(B \to \sigma B)]) \boxplus \widetilde{\mu}(\sigma\sigma B \Rightarrow s)\}\\
&\quad \wedge \{([\mu(S \to AB) \boxplus (\mu(A \to A\sigma) \wedge \mu(B \to \sigma B) \boxplus \mu(B \to \sigma)]\\
&\quad \wedge [\mu(S \to AB) \boxplus \mu(B \to \sigma) \boxplus \mu(A \to A\sigma)]) \boxplus \widetilde{\mu}(A\sigma\sigma \Rightarrow s)\}\\
&= \mu(S \to AB) \boxplus (\mu(A \to A\sigma) \wedge \mu(B \to \sigma B)) \boxplus \mu(A \to \sigma) \boxplus \mu(B \to \sigma)\\
&= a \boxplus (b \wedge c)
\end{aligned}$$

By the process described above, $G^C = (\{S, A, A', B, B'\}, \Sigma, S, \mu^C)$ where $\mu^C(S \to AB) = a$, $\mu^C(A \to AA') = b$, $\mu^C(B \to B'B) = c$, $\mu^C(A \to \sigma) = \mu^C(A' \to \sigma) = \mu^C(B \to \sigma) = \mu^C(B' \to \sigma) = \mathbf{0}$.

$$\begin{aligned}
|G^C|_w(s) &= \wedge_{m\geq 1}\left(\wedge_{\gamma_{m-1}\in\Gamma}\left(\cdots\left(\wedge_{\gamma_1\in\Gamma}\widetilde{\mu^C}(S\Rightarrow\gamma_1)\boxplus\widetilde{\mu^C}(\gamma_1\Rightarrow\gamma_2)\right)\cdots\right)\boxplus\widetilde{\mu^C}(\gamma_{m-1}\Rightarrow s)\right)\\
&= \wedge_{m\geq 1}(\wedge_{\gamma_{m-1}\in\Gamma}\left(\cdots\left(\wedge_{\gamma_2\in\Gamma}\mu^C(S\to AB)\boxplus\widetilde{\mu^C}(AB\Rightarrow\gamma_2)\boxplus\widetilde{\mu^C}(\gamma_2\Rightarrow\gamma_3)\right)\cdots\right)\boxplus\widetilde{\mu^C}(\gamma_{m-1}\Rightarrow s))\\
&= \wedge_{m\geq 1}(\wedge_{\gamma_{m-1}\in\Gamma}(\cdots([\mu^C(S\to AB)\boxplus\mu^C(A\to AA')\boxplus\widetilde{\mu^C}(AA'B\Rightarrow\gamma_3)]\\
&\quad\wedge[\mu^C(S\to AB)\boxplus\mu^C(A\to\sigma)\boxplus\widetilde{\mu^C}(\sigma B\Rightarrow\gamma_3)]\wedge[\mu^C(S\to AB)\boxplus\mu^C(B\to B'B)\boxplus\widetilde{\mu^C}(AB'B\Rightarrow\gamma_3)]\\
&\quad\wedge[\mu^C(S\to AB)\boxplus\mu^C(B\to\sigma)\boxplus\widetilde{\mu^C}(A\sigma\Rightarrow\gamma_3)])\cdots)\boxplus\widetilde{\mu^C}(\gamma_{m-1}\Rightarrow s))\\
&= \wedge_{m\geq 1}(\wedge_{\gamma_{m-1}\in\Gamma}(\cdots(\{([\mu^C(S\to AB)\boxplus\mu^C(A\to AA')\boxplus\mu^C(A'\to\sigma)]\\
&\quad\wedge[\mu^C(S\to AB)\boxplus\mu^C(B\to B'B)\boxplus\mu^C(B'\to\sigma)])\boxplus\widetilde{\mu^C}(A\sigma B\Rightarrow\gamma_4)\}\\
&\quad\wedge\{\mu^C(S\to AB)\boxplus\mu^C(A\to AA')\boxplus\mu^C(A\to\sigma)\boxplus\widetilde{\mu^C}(\sigma A'B\Rightarrow\gamma_4)\}\\
&\quad\wedge\{\mu^C(S\to AB)\boxplus\mu^C(A\to AA')\boxplus\mu^C(B\to\sigma)\boxplus\widetilde{\mu^C}(AA'\sigma\Rightarrow\gamma_4)\}\\
&\quad\wedge\{\mu^C(S\to AB)\boxplus\mu^C(A\to\sigma)\boxplus\mu^C(B\to B'B)\boxplus\widetilde{\mu^C}(\sigma B'B\Rightarrow\gamma_4)\}\\
&\quad\wedge\{\mu^C(S\to AB)\boxplus\mu^C(B\to B'B)\boxplus\mu^C(B\to\sigma)\boxplus\widetilde{\mu^C}(AB'\sigma\Rightarrow\gamma_4)\})\cdots)\boxplus\widetilde{\mu^C}(\gamma_{m-1}\Rightarrow s))\\
&= \{([\mu^C(S\to AB)\boxplus\mu^C(A\to AA')\boxplus\mu^C(A'\to\sigma)]\\
&\quad\wedge[\mu^C(S\to AB)\boxplus\mu^C(B\to B'B)\boxplus\mu^C(B'\to\sigma)])\boxplus\mu^C(A\to\sigma)\boxplus\widetilde{\mu^C}(\sigma\sigma B\Rightarrow s)\}\\
&\quad\wedge\{([\mu^C(S\to AB)\boxplus\mu^C(A\to AA')\boxplus\mu^C(A'\to\sigma)]\\
&\quad\wedge[\mu^C(S\to AB)\boxplus\mu^C(B\to B'B)\boxplus\mu^C(B'\to\sigma)])\boxplus\mu^C(B\to\sigma)\boxplus\widetilde{\mu^C}(A\sigma\sigma\Rightarrow s)\}\\
&\quad\wedge\{\mu^C(S\to AB)\boxplus\mu^C(A\to AA')\boxplus\mu^C(A\to\sigma)\boxplus(B\to\sigma)\boxplus\widetilde{\mu^C}(A'\to\sigma)\}\\
&\quad\wedge\{\mu^C(S\to AB)\boxplus\mu^C(A\to\sigma)\boxplus\mu^C(B\to B'B)\boxplus\mu^C(B\to\sigma)\boxplus\widetilde{\mu^C}(\sigma B'\sigma\Rightarrow s)\}\\
&= ([\mu(S\to AB)\boxplus\mu(A\to A\sigma)]\wedge[\mu(S\to AB)\boxplus\mu(B\to\sigma B)])\boxplus\mu(A\to\sigma)\boxplus\mu(B\to\sigma)\\
&= (a\boxplus b)\wedge(a\boxplus c)
\end{aligned}$$

Thus $|G|_w(s)=|G^C|_w(s)$ leads to $a\boxplus(b\wedge c)=(a\boxplus b)\wedge(a\boxplus c)$. Since a, b, c can be taken arbitrarily from $\mathcal{E}$, we get that $\boxplus$ distributes over $\wedge$ in $\mathcal{E}$. $\square$

Now, we rebuild $\mathcal{E}$-valued generalization of the Greibach normal form (GNF).

Lemma 7.3. *For any $G=(V,\Sigma,S,\mu)\in$CFG$(\mathcal{E},\Sigma)$, there is $G'=(V',\Sigma,S,\mu')\in$CFG$(\mathcal{E},\Sigma)$ such that $|G|_d(s)=|G'|_d(s)$ for any $s\in\Sigma^+$ and $\mu'(A\to\epsilon)=\mathbf{1}$ for any $A\in V'$.*

Proof. For any $p=A\to X_1X_2\cdots X_m\in PROD(G)$, any nonempty subset $I=\{i_1,i_2,\ldots,i_k\}(i_1<i_2<\cdots i_k)$ of $\{1,\ldots,m\}$ and any $\overline{\kappa}=(\kappa_1,\ldots,\kappa_k)\in D_G^+(X_{i_1},\epsilon)\times\cdots\times D_G^+(X_{i_k},\epsilon)$, introduce new variables $A_j^{(p,I,\overline{\kappa})}(j=1,\ldots,k+1)$. Note that when $X_{i_j}\in\Sigma$ for some $j\leq k$, then $A_j^{(p,I,\overline{\kappa})}$ is not defined, and $A_j^{(p,I,\overline{\kappa})}$ are different for distinct $(p,I,\overline{\kappa})$. Let

$$V'=V\cup\bigcup_{p,I,\overline{\kappa}}\{A_j^{(p,I,\overline{\kappa})}:j=1,\ldots,k+1\}$$

μ' is defined as:

$$\begin{aligned}
\mu'(A\to A_1^{(p,I,\overline{\kappa})}\cdots A_{k+1}^{(p,I,\overline{\kappa})}) &= \mu(p)\\
\mu'(A_1^{(p,I,\overline{\kappa})}\to X_1\cdots X_{i_1-1}) &= |\kappa_1|\\
&\cdots\\
\mu'(A_k^{(p,I,\overline{\kappa})}\to X_{i_{k-1}+1}\cdots X_{i_k-1}) &= |\kappa_k|\\
\mu'(A_{k+1}^{(p,I,\overline{\kappa})}\to X_{i_k+1}\cdots X_m) &= \mathbf{0}
\end{aligned}$$

and $\mu'(p)=\mu(p)$ for any $p\in PROD(G)-\{A\to\epsilon:A\in V\}$, $\mu'=\mathbf{1}$ for other cases.

Next we prove $|G|_d(s)=|G'|_d(s)$ for any $s\in\Sigma^+$. For any $\kappa=((\alpha_0=S,\alpha_1,\ldots,\alpha_n=s),(p_1,\ldots,p_n))\in D_G^+(S,s)$, suppose $p_j=A\to X_1\cdots X_m$ for some j. Let $I=\{i_1,i_2,\ldots,i_k\}$ be the set of indices of all variables which generate ϵ in κ. Denote $\beta=X_1\cdots X_{i_1-1}X_{i_1+1}\cdots X_m$, that is, the sequence obtained by remove $\{X_{i_j}\}$ from $X_1\cdots X_m$. Assume that

$$\begin{aligned}
&\kappa_{i_1}=((X_{i_1},\ldots,\epsilon),(p_{j+1},\ldots,p_{j_1}))\in D_G^+(X_{i_1},\epsilon)\\
&\kappa_{i_2}=((X_{i_2},\ldots,\epsilon),(p_{j_1+1},\ldots,p_{j_2}))\in D_G^+(X_{i_2},\epsilon)\\
&\cdots\\
&\kappa_{i_k}=((X_{i_k},\ldots,\epsilon),(p_{j_{k-1}+1},\ldots,p_{j_k}))\in D_G^+(X_{i_k},\epsilon)
\end{aligned}$$

where $p_j, \ldots, p_{j_1}, \ldots, p_{j_2}, \ldots, p_{j_k}$ is subsequence of productions in κ. The derivation κ could be written as

$$\kappa = ((S, \ldots, \gamma_1 A\gamma_2, \gamma_1 X_1 \cdots, X_m\gamma_2, \gamma_1\beta\gamma_2, \ldots, s), (\cdots, p_j, p_{j+1}, \ldots, p_{j_k}, \ldots))$$

Define $\overline{\kappa} = (\kappa_{i_1}, \ldots, \kappa_{i_k})$. There is $\kappa' = ((S, \ldots, \gamma_1 A\gamma_2, \gamma_1\beta\gamma_2, \ldots, s), (\cdots, p'_j, \ldots, p'_{j+k}, \ldots)) \in D^+_{G'}(S, s)$, where

$$p'_j = A \to A_1^{(p,I,\overline{\kappa})} \cdots A_{k+1}^{(p,I,\overline{\kappa})}$$
$$p'_{j+1} = A_1^{(p,I,\overline{\kappa})} \to X_1 \cdots X_{i_1-1}$$
$$\cdots$$
$$p'_{j+k+1} = A_{k+1}^{(p,I,\overline{\kappa})} \to X_{i_k+1} \cdots X_m$$

Note that the ϵ-productions following the derivations of $\{X_{i_j}\}$ do not exist in κ' and $|p_j| \boxplus \cdots \boxplus |p_{j_k}| = |p'_j| \boxplus \cdots \boxplus |p'_{j+k+1}|$. In this way we could find a $\kappa'' \in D^+_{G'}(S, s)$ satisfying $|\kappa''| = |\kappa|$.

Conversely replace the production family $\{p'_j, \ldots, p'_{j+k+1}\}$ with $\{p_j, \ldots, p_{j_k}\}$ in some $\kappa'' \in D^+_{G'}(S, s)$ we could get $\kappa \in D^+_G(S, s)$ satisfying $|\kappa| = |\kappa''|$. □

Lemma 7.4. *For any $G = (V, \Sigma, S, \mu) \in$CFG$(\mathcal{E}, \Sigma)$,there exists $G' = (V', \Sigma, S, \mu') \in$CFG$(\mathcal{E}, \Sigma)$ such that $|G|_d = |G'|_d$ and $\mu'(A \to B) = \mathbf{1}$ for any $A, B \in V'$.*

Proof. Let $PROD_{unit}(G) = \{A \to B : A, B \in V\}$ be the set of unit productions of G_1. For any $\kappa = ((A, \alpha_1, \ldots, B), (p_1, \ldots, p_n)) \in D^+_G(A, B)$, we could assume $\alpha_i \in V$ since $\mu(A \to \epsilon) = \mathbf{1}$ for all $A \in V$. Furthermore we could suppose that $A, \alpha_1, \ldots, \alpha_n, B$ are all different. Now introduce new variables $[A, B, \kappa]$ for each $A, B \in V$ and $\kappa \in D^+_G(A, B)$, define $V' = V \cup \{[A, B, \kappa] : A, B \in V, \kappa \in D^+_G(A, B)\}$. Since $\{[A, B, \kappa] : A, B \in V, \kappa \in D^+_G(A, B)\}$ is finite by the assumption that variables do not repeat in each κ, then V' is finite.

Denote $V_A = \{A\} \cup \{[A, B, \kappa] : B \in V, \kappa \in D^+_G(A, B)\}$ and $V_\sigma = \{\sigma\}$. Define μ' as follows:

(i) $\mu'(A \to X_1 \cdots X_m) = \mu(A \to Y_1 \cdots Y_m)$ if $X_i \in V_{Y_i}$, $A \to Y_1 \cdots Y_m \in PROD(G) - PROD_{unit}(G)$.
(ii) $\mu'([A, B, \kappa] \to \alpha) = \mu(B \to \alpha) \boxplus |\kappa|$ if $B \to \alpha \in PROD(G) - PROD_{unit}(G)$.
(iii) $\mu'(p) = \mathbf{1}$ for others.

Let $\kappa = ((S, \alpha_1, \ldots, s), (p_1, \ldots, p_n)) \in D^+_G(S, s)$. If there are unit productions in κ, we could assume $p_h = A \to \gamma_1 B_1\gamma_2 \notin PROD_{unit}(G)$, $p_{h+i} = B_i \to B_{i+1}$ $(i = 1, \ldots, l-1)$ and $p_{h+l} = B_l \to \alpha \notin PROD_{unit}(G)$ by Remark 7.1. Then there is a derivation $\kappa' \in D^+_{G'}(S, s)$ such that the unit productions p_{h+i} are removed from κ and some nonunit productions are added. Denote $\kappa_1 = ((B_1, B_2, \ldots, B_l), (p_{h+1}, \ldots, p_{h+l-1}))$, then new variable $[B_1, B_l, \kappa_1]$ is introduced. Construct κ' as

$$\kappa' = ((S, \ldots, \alpha_1[B_1, B_l, \kappa_1]\alpha_2, \alpha_1\alpha\alpha_2, \ldots, s), (\ldots, A \to \gamma_1[B_1, B_l, \kappa_1]\gamma_2, [B_1, B_l, \kappa_1] \to \alpha, \ldots))$$

Easy to see $|\kappa'| = |\kappa|$. By this means we could find $\kappa'' \in D^+_{G'}(S, s)$ with no unit productions satisfying $|\kappa''| = |\kappa|$ for any $\kappa \in D^+_G(S, s)$.

Conversely if execute the above process in the opposite direction we could get $\kappa \in D^+_G(S, s)$ such that $|\kappa| = |\kappa''|$ for any $\kappa'' \in D^+_{G'}(S, s)$. □

Lemmata 7.3 and 7.4 imply the following conclusions:

Corollary 7.5. *For any $G = (V, \Sigma, S, \mu) \in$CFG$(\mathcal{E}, \Sigma)$, there is $G' = (V', \Sigma, S, \mu') \in$CFG$(\mathcal{E}, \Sigma)$ such that $|G|_d = |G'|_d$ and satisfying:*

(i) $\mu'(A \to \epsilon) = \mathbf{1}$ *for any* $A \in V' - \{S\}$.
(ii) $\mu'(A \to B) = \mathbf{1}$ *for any* $A, B \in V'$.

Proof. By Lemmas 7.3 and 7.4 we get $G'' = (V', \Sigma, S, \mu'')$ such that $|G''|_d(s) = |G|_d(s)$ for $s \in \Sigma^+$, and $\mu''(A \to \epsilon) = \mathbf{1}$ for all $A \in V''$, $\mu''(A \to B) = \mathbf{1}$ for all $A, B \in V''$. Define $G' = (V', \Sigma, S, \mu')$ where $\mu'(S \to \epsilon) = |G|_d(\epsilon)$ and $\mu' = \mu''$ for others, which is the grammar we need. □

Corollary 7.6. *If $G \in$CNF$(\mathcal{E}, \Sigma)$ in Corollary 7.5, then also $G' \in$CNF$(\mathcal{E}, \Sigma)$.*

Lemma 7.7. *Let $G = (V, \Sigma, S, \mu) \in$CFG$(\mathcal{E}, \Sigma)$. Suppose $\{A \to A\alpha_i\}_i$ are all the productions in supp(μ) that begin with A and the leftmost symbol in the right side is A. Then there exists $G' = (V', \Sigma, S, \mu')$ such that $|G'|_d = |G|_d$ and $\{A \to A\alpha_i\}_i \not\subset$ supp(μ').*

Proof. Let $\{A \to \beta_j\}_j$ be all the productions in $supp(\mu)$ that the leftmost symbol in the right is not A. According to the techniques in classical automata theory, introduce a new variable $A' \notin V$, denote $V' = \{A'\} \cup V$, define

$$\mu'(A \to \beta_j) = \mu'(A \to \beta_j A') = \mu(A \to \beta_j)$$
$$\mu'(A' \to \alpha_i) = \mu(A' \to \alpha_i A') = \mu(A \to A\alpha_i),$$

and $\mu'(p) = \mu(p)$ if $p \in PROD(G) - \{A \to A\alpha_i\}_i$, $\mu'(p) = \mathbf{1}$ for other cases. It is easy to check that $|G'|_d = |G|_d$. □

Remark 7.3. Let $G = (V, \Sigma, S, \mu) \in$CFG$(\mathcal{E}, \Sigma)$. For $\forall A \neq B \in V$, suppose that $\{A \to B\alpha_i\}_i$ are all productions in $supp(\mu)$ that begin with A and the leftmost symbol in the right side is B, and $\{B \to \beta_j\}_j$ are all productions in $supp(\mu)$ that begin with B. If $A \to \beta_j\alpha_i \notin supp(\mu)$ for $\forall i, j$, there exists $G' = (V', \Sigma, S, \mu')$ such that $|G'|_d = |G|_d$ and $\mu'(A \to B\alpha_i) = \mathbf{1}$ for $\forall i$. To see that, just define $\mu'(A \to \beta_j\alpha_i) = \mu(A \to B\alpha_i) \boxplus \mu(B \to \beta_j)$ and $\mu'(A \to B\alpha_i) = \mathbf{1}$.

The following theorem establishes the generalization of Greibach normal form theorem in unsharp quantum logic. It is easy to see that the distributive law is not required in depth-first mode, whereas it is needed in the width-first mode.

Theorem 7.8. (i) $|G|_d = |G^G|_d$ *for any* $G \in CFG(\mathcal{E}, \Sigma)$.
(ii) $|G|_w = |G^G|_w$ *for any* $G \in CFG(\mathcal{E}, \Sigma)$ *iff* $a \boxplus (b \wedge c) = (a \boxplus b) \wedge (a \boxplus c)$ *for any* $a, b, c \in \mathcal{E}$.

Proof. (i) By Theorem 7.2 and Corollary 7.6, for any $G \in$ CFG$(\mathcal{E}, \Sigma)$, there exists $G' \in$CNF$(\mathcal{E}, \Sigma)$ satisfying (i) and (ii) in Corollary 7.5. Now, by Lemma 7.7 and Remark 7.3, similar to that in classical automata theory, we can construct a $G'^G \in$GNF$(\mathcal{E}, \Sigma)$ such that $|G|_d = |G'|_d = |G'^G|_d$. We omit the details here.

(ii) By Theorem 7.1 and (i), we only need to prove "only if" part. If $|G|_w = |G^G|_w$ for any $G \in$ CFG, then for any $G \in$ CNG, $|G|_w = |G^G|_w$. For any $a, b, c \in \mathcal{E}$ we find a grammar $G = (V, \Sigma, S, \mu)$ in Chomsky normal form as

- $V = \{S, A_1, A_2, A_3\}$.
- let $\sigma \in \Sigma$, $\mu(S \to A_1A_2) = a$, $\mu(A_1 \to A_1A_3) = b$, $\mu(A_1 \to \sigma) = \mathbf{0}$, $\mu(A_2 \to A_3A_2) = c$, $\mu(A_2 \to \sigma) = \mathbf{0}$, $\mu(A_3 \to \sigma) = \mathbf{0}$, and $\mu = \mathbf{1}$ for other cases.

Given $s = \sigma\sigma\sigma$, then

$$
\begin{aligned}
|G|_w(s) &= \bigwedge_{\gamma_4}\left(\bigwedge_{\gamma_3}\left(\bigwedge_{\gamma_2}\left(\bigwedge_{\gamma_1}\widetilde{\mu}(S \Rightarrow \gamma_1) \boxplus \widetilde{\mu}(\gamma_1 \Rightarrow \gamma_2)\right) \boxplus \widetilde{\mu}(\gamma_2 \Rightarrow \gamma_3)\right) \boxplus \widetilde{\mu}(\gamma_3 \Rightarrow \gamma_4)\right) \boxplus \widetilde{\mu}(\gamma_4 \Rightarrow s) \\
&= \left[\left(\bigwedge_{\gamma_3}\left(\bigwedge_{\gamma_2}\widetilde{\mu}(S \Rightarrow A_1A_2) \boxplus \widetilde{\mu}(A_1A_2 \Rightarrow \gamma_2) \boxplus \widetilde{\mu}(\gamma_2 \Rightarrow \gamma_3)\right) \boxplus \widetilde{\mu}(\gamma_3 \Rightarrow A_1\sigma\sigma)\right) \boxplus \widetilde{\mu}(A_1\sigma\sigma \Rightarrow s)\right] \\
&\quad \bigwedge\left[\left(\bigwedge_{\gamma_3}\left(\bigwedge_{\gamma_2}\widetilde{\mu}(S \Rightarrow A_1A_2) \boxplus \widetilde{\mu}(A_1A_2 \Rightarrow \gamma_2) \boxplus \widetilde{\mu}(\gamma_2 \Rightarrow \gamma_3)\right) \boxplus \widetilde{\mu}(\gamma_3 \Rightarrow \sigma A_3\sigma)\right) \boxplus \widetilde{\mu}(\sigma A_3\sigma \Rightarrow s)\right] \\
&\quad \bigwedge\left[\left(\bigwedge_{\gamma_3}\left(\bigwedge_{\gamma_2}\widetilde{\mu}(S \Rightarrow A_1A_2) \boxplus \widetilde{\mu}(A_1A_2 \Rightarrow \gamma_2) \boxplus \widetilde{\mu}(\gamma_2 \Rightarrow \gamma_3)\right) \boxplus \widetilde{\mu}(\gamma_3 \Rightarrow \sigma\sigma A_2)\right) \boxplus \widetilde{\mu}(\sigma\sigma A_2 \Rightarrow s)\right] \\
&= \left\{\left[\left(\left(\bigwedge_{\gamma_2}\widetilde{\mu}(S \Rightarrow A_1A_2) \boxplus \widetilde{\mu}(A_1A_2 \Rightarrow \gamma_2) \boxplus \widetilde{\mu}(\gamma_2 \Rightarrow A_1A_3\sigma)\right) \boxplus \widetilde{\mu}(A_1A_3\sigma \Rightarrow A_1\sigma\sigma)\right)\right.\right. \\
&\quad \left.\left.\bigwedge\left(\left(\bigwedge_{\gamma_2}\widetilde{\mu}(S \Rightarrow A_1A_2) \boxplus \widetilde{\mu}(A_1A_2 \Rightarrow \gamma_2)\widetilde{\mu}(\gamma_2 \Rightarrow A_1\sigma A_2)\right) \boxplus \widetilde{\mu}(A_1\sigma A_2 \Rightarrow A_1\sigma\sigma)\right)\right] \boxplus \widetilde{\mu}(A_1\sigma\sigma \Rightarrow s)\right\} \\
&\quad \bigwedge\left\{\left[\left(\left(\bigwedge_{\gamma_2}\widetilde{\mu}(S \Rightarrow A_1A_2) \boxplus \widetilde{\mu}(A_1A_2 \Rightarrow \gamma_2) \boxplus \widetilde{\mu}(\gamma_2 \Rightarrow A_1A_3\sigma)\right) \boxplus \widetilde{\mu}(A_1A_3\sigma \Rightarrow \sigma A_3\sigma)\right)\right.\right. \\
&\quad \left.\left.\bigwedge\left(\left(\bigwedge_{\gamma_2}\widetilde{\mu}(S \Rightarrow A_1A_2) \boxplus \widetilde{\mu}(A_1A_2 \Rightarrow \gamma_2)\widetilde{\mu}(\gamma_2 \Rightarrow \sigma A_3A_2)\right) \boxplus \widetilde{\mu}(\sigma A_3A_2 \Rightarrow \sigma A_3\sigma)\right)\right] \boxplus \widetilde{\mu}(\sigma A_3\sigma \Rightarrow s)\right\} \\
&\quad \bigwedge\left\{\left[\left(\left(\bigwedge_{\gamma_2}\widetilde{\mu}(S \Rightarrow A_1A_2) \boxplus \widetilde{\mu}(A_1A_2 \Rightarrow \gamma_2) \boxplus \widetilde{\mu}(\gamma_2 \Rightarrow \sigma A_3A_2)\right) \boxplus \widetilde{\mu}(\sigma A_3A_2 \Rightarrow \sigma\sigma A_2)\right)\right.\right. \\
&\quad \left.\left.\bigwedge\left(\left(\bigwedge_{\gamma_2}\widetilde{\mu}(S \Rightarrow A_1A_2) \boxplus \widetilde{\mu}(A_1A_2 \Rightarrow \gamma_2)\widetilde{\mu}(\gamma_2 \Rightarrow A_1\sigma A_2)\right) \boxplus \widetilde{\mu}(A_1\sigma A_2 \Rightarrow \sigma\sigma A_2)\right)\right] \boxplus \widetilde{\mu}(\sigma\sigma A_2 \Rightarrow s)\right\} \\
&= \mu(S \to A_1A_2) \boxplus \widetilde{\mu}(A_1A_2 \Rightarrow A_1A_3A_2) \boxplus \mu(A_1 \to \sigma) \boxplus \mu(A_2 \to \sigma) \boxplus \mu(A_3 \to \sigma) \\
&= a \boxplus (b \wedge c)
\end{aligned}
$$

On the other hand, the corresponding grammar in Greibach normal form is $G^G = (V', \Sigma, S, \mu')$:

- $V' = \{S, A_1, A_2, A_3, B\}$.
- μ' is defined as

$$\mu'(S \to \sigma A_2) = \mu(A_1 \to \sigma) \boxplus \mu(S \to A_1A_2) = a$$
$$\mu'(S \to \sigma BA_2) = \mu(S \to A_1A_2) \boxplus \mu(A_1 \to \sigma) = a$$
$$\mu'(A_1 \to \sigma) = \mu(A_1 \to \sigma) = \mathbf{0}$$
$$\mu'(A_1 \to \sigma B) = \mu(A_1 \to \sigma) = \mathbf{0}$$
$$\mu'(A_2 \to \sigma) = \mu(A_2 \to \sigma) = \mathbf{0}$$
$$\mu'(A_2 \to \sigma A_2) = \mu(A_3 \to \sigma) \boxplus \mu(A_2 \to A_3A_2) = c$$
$$\mu'(A_3 \to \sigma) = \mu(A_3 \to \sigma) = \mathbf{0}$$
$$\mu'(B \to \sigma) = \mu(A_1 \to A_1A_3) \boxplus \mu(A_3 \to \sigma) = b$$
$$\mu'(B \to \sigma B) = \mu(A_1 \to A_1A_3) \boxplus \mu(A_3 \to \sigma) = b$$

and $\mu' = \mathbf{1}$ for other cases.

Then

$$\begin{aligned}
|G^G|_w(s) &= \bigwedge_{\gamma_2}\left(\bigwedge_{\gamma_1}\widetilde{\mu}'(S \Rightarrow \gamma_1) \boxplus \widetilde{\mu}'(\gamma_1 \Rightarrow \gamma_2)\right) \boxplus \widetilde{\mu}'(\gamma_2 \Rightarrow s)\\
&= \left[\left(\bigwedge_{\gamma_1}\widetilde{\mu}'(S \Rightarrow \gamma_1) \boxplus \widetilde{\mu}'(\gamma_1 \Rightarrow \sigma\sigma A_2)\right) \boxplus \widetilde{\mu}'(\sigma\sigma A_2 \Rightarrow s)\right]\\
&\quad \bigwedge\left[\left(\bigwedge_{\gamma_1}\widetilde{\mu}'(S \Rightarrow \gamma_1) \boxplus \widetilde{\mu}'(\gamma_1 \Rightarrow \sigma B\sigma)\right) \boxplus \widetilde{\mu}'(\sigma B\sigma \Rightarrow s)\right]\\
&= \big[\big((\widetilde{\mu}'(S \Rightarrow \sigma A_2) \boxplus \widetilde{\mu}'(\sigma A_2 \Rightarrow \sigma\sigma A_2)) \wedge (\widetilde{\mu}'(S \Rightarrow \sigma BA_2) \boxplus \widetilde{\mu}'(\sigma BA_2 \Rightarrow \sigma\sigma A_2))\big) \boxplus \widetilde{\mu}'(\sigma\sigma A_2 \Rightarrow s)\big]\\
&\quad \bigwedge\big[\widetilde{\mu}'(S \Rightarrow \sigma BA_2) \boxplus \widetilde{\mu}'(\sigma BA_2 \Rightarrow \sigma B\sigma) \boxplus \widetilde{\mu}'(\sigma B\sigma \Rightarrow s)\big]\\
&= \big[\big(\big(\mu'(S \to \sigma A_2) \boxplus \mu'(A_2 \to \sigma A_2)\big) \wedge \big(\mu'(S \to \sigma BA_2) \boxplus \mu'(B \to \sigma)\big)\big) \boxplus \mu'(A_2 \to \sigma)\big]\\
&\quad \bigwedge\big[\mu'(S \to \sigma BA_2) \boxplus \mu'(A_2 \to \sigma) \boxplus \mu'(B \to \sigma)\big]\\
&= [(a \boxplus c) \wedge (a \boxplus b) \boxplus \mathbf{0}] \wedge [a \boxplus \mathbf{0} \boxplus b]\\
&= (a \boxplus b) \wedge (a \boxplus c)
\end{aligned}$$

Thus $|G|_w = |G^G|_w$ leads to $(a \boxplus b) \wedge (a \boxplus c) = a \boxplus (b \wedge c)$. Since a, b, c are taken arbitrarily from $\mathcal{E}$, we get that $\boxplus$ distributes over $\wedge$. □

The following two results show that various variants of $\mathcal{E}$-valued context free grammar do not preserve the recognizability of language in general in width-first principle.

Corollary 7.9. *$|G|_w = |(G^C)^G|_w$ for any $G \in CFG(\mathcal{E}, \Sigma)$ iff $a \boxplus (b \wedge c) = (a \boxplus b) \wedge (a \boxplus c)$ for any $a, b, c \in \mathcal{E}$.*

Proof. "If part" If the distributive law is true, then $|G|_w = |(G^C)|_w = |(G^C)^G|_w$ by Theorems 8.1 and 8.2(ii).

"Only if part". Suppose the distributive law is not true. Then there are $a, b, c \in \mathcal{E}$ such that $(a \boxplus b) \wedge (a \boxplus c) \neq a \boxplus (b \wedge c)$. Let G be a Chomsky grammar as that in proof of Theorems 7.2(ii) and 7.8(ii), then $G = G^C$. Similarly to (ii), we can construct corresponding Greibach normal form G^G. By a calculation, we see that $|G|_w \neq |G^G|_w$. □

Theorem 7.10. *$|G|_w = |(G^G)^C|_w$ for any $G \in CFG(\mathcal{E}, \Sigma)$ iff $a \boxplus (b \wedge c) = (a \boxplus b) \wedge (a \boxplus c)$ for any $a, b, c \in \mathcal{E}$.*

Proof. "If part" If the distributive law is true, then $|G|_w = |G^C|_w = |(G^G)^C|_w$.

"Only if" part: For any $a, b, c \in \mathcal{E}$, consider $G = (V, \Sigma, S, \mu)$ in Greibach normal form as $V = \{S, A, B, C\}$, and for some $\sigma \in \Sigma$, μ is defined as $\mu(S \to \sigma A) = a$, $\mu(S \to \sigma BC) = b$, $\mu(C \to \sigma) = c$, $\mu(A \to \sigma C) = \mathbf{0}$, $\mu(B \to \sigma) = \mathbf{0}$ and $\mu = \mathbf{1}$ for the rest.

Given $s = \sigma\sigma\sigma$,

$$\begin{aligned}
|G|_w(s) &= \bigwedge_{\gamma_2}\left(\bigwedge_{\gamma_1}\widetilde{\mu}(S \Rightarrow \gamma_1) \boxplus \widetilde{\mu}(\gamma_1 \Rightarrow \gamma_2)\right) \boxplus \widetilde{\mu}(\gamma_2 \Rightarrow s)\\
&= \left[\left(\bigwedge_{\gamma_1}\widetilde{\mu}(S \Rightarrow \gamma_1) \boxplus \widetilde{\mu}(\gamma_1 \Rightarrow \sigma\sigma C)\right) \boxplus \widetilde{\mu}(\sigma\sigma C \Rightarrow s)\right]
\end{aligned}$$

$$\bigwedge\left[\left(\bigwedge_{\gamma_1}\widetilde{\mu}(S\Rightarrow\gamma_1)\boxplus\widetilde{\mu}(\gamma_1\Rightarrow\sigma B\sigma)\right)\boxplus\widetilde{\mu}(\sigma B\sigma\Rightarrow s)\right]$$
$$=[((\mu(S\to\sigma A)\boxplus\mu(A\to\sigma C))\wedge(\mu(S\to\sigma BC)\boxplus\mu(B\to\sigma)))\boxplus\mu(C\Rightarrow\sigma)]$$
$$\wedge\ [\mu(S\to\sigma BC)\boxplus\mu(C\to\sigma)\boxplus\mu(B\to\sigma)]$$
$$=(a\wedge b)\boxplus c$$

On the other hand, the corresponding grammar G^G is exactly G, and $(G^G)^C=(V',\Sigma,S,\mu')$:

- $V'=\{S,A,B,C,X_1,X_2,X_3,X_4\}$.
- μ' is constructed as

$$\mu'(S\to X_1A)=\mu(S\to\sigma A)=a,\ \mu'(X_1\to\sigma)=\mathbf{0}$$
$$\mu'(S\to X_2X_3)=\mu(S\to\sigma BC)=b,\ \mu'(X_2\to\sigma)=\mathbf{0},\ \mu'(X_3\to BC)=\mathbf{0}$$
$$\mu'(A\to X_4C)=\mu(A\to\sigma C)=\mathbf{0},\ \mu'(X_4\to\sigma)=\mathbf{0}$$
$$\mu'(B\to\sigma)=\mathbf{0},\ \mu'(C\to\sigma)=\mu(C\to\sigma)=c$$

for some $\sigma\in\Sigma$.

Then

$$|(G^G)^C|_w(s)=\bigwedge_{\gamma_4}\left(\bigwedge_{\gamma_3}\left(\bigwedge_{\gamma_2}\left(\bigwedge_{\gamma_1}\widetilde{\mu}'(S\Rightarrow\gamma_1)\boxplus\widetilde{\mu}'(\gamma_1\Rightarrow\gamma_2)\right)\boxplus\widetilde{\mu}'(\gamma_2\Rightarrow\gamma_3)\right)\boxplus\widetilde{\mu}'(\gamma_3\Rightarrow\gamma_4)\right)\boxplus\widetilde{\mu}'(\gamma_4\Rightarrow s)$$
$$=\left[\left(\bigwedge_{\gamma_3}\left(\bigwedge_{\gamma_2}\left(\bigwedge_{\gamma_1}\widetilde{\mu}'(S\Rightarrow\gamma_1)\boxplus\widetilde{\mu}'(\gamma_1\Rightarrow\gamma_2)\right)\boxplus\widetilde{\mu}'(\gamma_2\Rightarrow\gamma_3)\right)\boxplus\widetilde{\mu}'(\gamma_3\Rightarrow X_1\sigma\sigma)\right)\boxplus\widetilde{\mu}'(X_1\sigma\sigma\Rightarrow s)\right]$$
$$\bigwedge\left[\left(\bigwedge_{\gamma_3}\left(\bigwedge_{\gamma_2}\left(\bigwedge_{\gamma_1}\widetilde{\mu}'(S\Rightarrow\gamma_1)\boxplus\widetilde{\mu}'(\gamma_1\Rightarrow\gamma_2)\right)\boxplus\widetilde{\mu}'(\gamma_2\Rightarrow\gamma_3)\right)\boxplus\widetilde{\mu}'(\gamma_3\Rightarrow\sigma X_4\sigma)\right)\boxplus\widetilde{\mu}'(\sigma X_4\sigma\Rightarrow s)\right]$$
$$\bigwedge\left[\left(\bigwedge_{\gamma_3}\left(\bigwedge_{\gamma_2}\left(\bigwedge_{\gamma_1}\widetilde{\mu}'(S\Rightarrow\gamma_1)\boxplus\widetilde{\mu}'(\gamma_1\Rightarrow\gamma_2)\right)\boxplus\widetilde{\mu}'(\gamma_2\Rightarrow\gamma_3)\right)\boxplus\widetilde{\mu}'(\gamma_3\Rightarrow X_2\sigma\sigma)\right)\boxplus\widetilde{\mu}'(X_2\sigma\sigma\Rightarrow s)\right]$$
$$\bigwedge\left[\left(\bigwedge_{\gamma_3}\left(\bigwedge_{\gamma_2}\left(\bigwedge_{\gamma_1}\widetilde{\mu}'(S\Rightarrow\gamma_1)\boxplus\widetilde{\mu}'(\gamma_1\Rightarrow\gamma_2)\right)\boxplus\widetilde{\mu}'(\gamma_2\Rightarrow\gamma_3)\right)\boxplus\widetilde{\mu}'(\gamma_3\Rightarrow\sigma B\sigma)\right)\boxplus\widetilde{\mu}'(\sigma B\sigma\Rightarrow s)\right]$$
$$\bigwedge\left[\left(\bigwedge_{\gamma_3}\left(\bigwedge_{\gamma_2}\left(\bigwedge_{\gamma_1}\widetilde{\mu}'(S\Rightarrow\gamma_1)\boxplus\widetilde{\mu}'(\gamma_1\Rightarrow\gamma_2)\right)\boxplus\widetilde{\mu}'(\gamma_2\Rightarrow\gamma_3)\right)\boxplus\widetilde{\mu}'(\gamma_3\Rightarrow\sigma\sigma C)\right)\boxplus\widetilde{\mu}'(\sigma\sigma C\Rightarrow s)\right]$$
$$=(\mu(S\to\sigma A)\boxplus\mu(A\to\sigma C)\boxplus\mu(C\to\sigma))\wedge(\mu(S\to\sigma BC)\boxplus\mu(B\to\sigma)\boxplus(c\to\sigma)$$
$$=(a\boxplus c)\wedge(b\boxplus c)$$

Thus $|G|_w=|(G^G)^C|_w$ results in $(a\wedge b)\boxplus c=(a\boxplus c)\wedge(b\boxplus c)$. $\square$

8. $\mathcal{E}$-valued pushdown automata

Pushdown automata form another important class of finite state machines. In this section, we reformulate the theory of pushdown automata into the framework of unsharp quantum logic and to observe the essential differences between quantum and classical theory of pushdown automata.

Definition 8.1. An $\mathcal{E}$-valued pushdown automaton ($\mathcal{E}$PDA) M consists of seven components $M=(Q,\Sigma,\Lambda,\delta,q_0,Z_0,T)$:

- Q is a finite set of states.
- Σ is the input alphabet.
- Λ is the stack alphabet.
- $q_0\in Q$ is the initial state.
- $Z_0\in\Lambda$ is a special stack symbol called the start symbol.
- $T:Q\longrightarrow\mathcal{E}$, is the final state function.
- $\delta:Rule(M)=(Q\times\Sigma_\epsilon\times\Lambda)\times(Q\times\Lambda^*)\longrightarrow\mathcal{E}$, is the transition function. The support $supp(\delta)=\{((q,w,Z),(p,\alpha)):\delta((q,w,Z),(p,\alpha))\neq\mathbf{1}\}$ is finite.

Intuitively, for any $r = ((q, w, Z), (p, \sigma)) \in Rule(M)$, $\delta(r)$ stands for the truth value of the proposition that M, in the current state q and the top symbol Z on the stack, with input symbol ω, can enter state p and replace Z with $\sigma \in \Lambda^*$ and advance one symbol if $\omega \neq \varepsilon$.

Since the distributivity of $\boxplus$ over $\wedge$ in $\mathcal{E}$ does not hold, there are again two kinds of methods to define the language accepted by an $\mathcal{E}$-valued pushdown automata. First, we give the definition from the depth-first mode.

For $\mathcal{E}$-valued PDA, similar to classical PDA, we use configuration to define the instantaneous description of a M and the language accepted by M. Configuration is a triple $c = (q, w, \gamma) \in Q \times \Sigma^* \times \Lambda^*$, where q is the current state, w is the sequences of the remaining input characters, γ is the sequence of symbols in the stack. Let $CON(\mathcal{M})$ denote the set of configurations, that is $CON(\mathcal{M}) = Q \times \Sigma^* \times \Lambda^*$. The next configuration relation $\vdash_M$ is defined as follows: for any $p, q \in Q$, $\sigma \in \Sigma \cup \{\varepsilon\}$, $\omega \in \Sigma^*$, $Z \in \Lambda$ and $\alpha, \beta \in \Lambda^*$, if $((p, \sigma, Z), (q, \beta)) \in \delta$ then we have $c_1 = (p, \sigma w, Z\gamma) \vdash_M c_2(q, w, \beta\gamma)$. In this case, we say that $r = ((p, \sigma, Z), (q, \beta))$ is compatible with the next configuration relation $c_1 \vdash_M c_2$ and denoted as $r_{c_1 \vdash_M c_2}$. Note that $r_{c_1 \vdash_M c_2}$ is unique if exists for any c_1 and c_2. The $\mathcal{E}$-value of this transition is defined to be $|c_1 \vdash_M c_2| = \delta((p, \sigma, Z), (q, \beta))$.

A path P of M is a sequence of configurations $P = (c_0, \ldots, c_n)$ where $c_{i-1} \vdash_M c_i$ for $n \geq 1$, $i = 1, \ldots, n$. The length of P is denoted as $l(P) = n$. The set of all paths of length n is denoted by $Path_n(M)$. All paths of M is denoted by $Path(M) = \bigcup_{n \geq 1} Path_n(M)$. The $\mathcal{E}$-value of path P is defined to be $|P| = |c_0 \vdash_M c_1| \boxplus \cdots \boxplus |c_{n-1} \vdash_M c_n|$. We denote $b(P) = c_0$, $e(P) = c_n$. If $P_1 = (c_0, \ldots, c_n)$, $P_2 = (c_n, \ldots, c_m)$, define the concatenation of P_1, P_2 to be $P = P_1P_2 = (c_0, \ldots, c_m)$. Obviously $|P| = |P_1| \boxplus |P_2|$.

In the following, we denote PDA$(\mathcal{E}, \Sigma)$ to be the set of all $\mathcal{E}$PDA over Σ.

Definition 8.2. The $\mathcal{E}$-valued language accepted by an $\mathcal{E}$PDA $M = (Q, \Sigma, \Lambda, \delta, q_0, Z_0, T)$ by final state in a depth-first mode, denoted by $|M|_d^f$, is defined as:

$$|M|_d^f(s) = \bigwedge\{|P| \boxplus T(q) : P \in Path(M), b(P) = (q_0, s, Z_0), e(P) = (q, \epsilon, \gamma)\} \tag{14}$$

for any $s \in \Sigma^*$.

Definition 8.3. The $\mathcal{E}$-valued language accepted by an $\mathcal{E}$PDA $M = (Q, \Sigma, \Lambda, \delta, q_0, Z_0, T)$ by empty stack in a depth-first mode, denoted by $|M|_d^e$, is defined as:

$$|M|_d^e(s) = \bigwedge\{|P| : P \in Path(M), b(P) = (q_0, s, Z_0), e(P) = (q, \epsilon, \epsilon)\} \tag{15}$$

for any $s \in \Sigma^*$.

The following result shows that the two concepts of language accepted by final state and empty stack are equivalent in the depth-first mode.

Theorem 8.1. *For any $\mathcal{E}$PDA M_1, there is an $\mathcal{E}$PDA M_2 such that $|M_1|_d^f = |M_2|_d^e$. Conversely, for any $\mathcal{E}$PDA M_1, there is an $\mathcal{E}$PDA M_2 such that $|M_1|_d^e = |M_2|_d^f$.*

Proof. The first part. For any $\mathcal{E}$PDA M_1, construct M_2 with the same technique as in classical automata theory. Assume $M_1 = (Q_1, \Sigma, \Lambda_1, \delta_1, q_1, Z_1, T_1)$, define $M_2 = (Q_2, \Sigma, \Lambda_2, \delta_2, q_2, Z_2, T_2)$ as:

(1) $Q_2 = Q_1 \cup \{q_0, q_2\}$ where $q_0, q_2 \notin Q_1$.
(2) $\Lambda_2 = \Lambda_1 \cup \{Z_2\}$ where $Z_2 \notin \Lambda_1$.
(3) δ_2:
   (a) $\delta_2((q_2, \epsilon, Z_2), (q_1, Z_1Z_2)) = \mathbf{0}$.
   (b) $\delta_2|_{\Lambda_1} = \delta_1$.
   (c) $\delta_2((q, \epsilon, A), (q_0, A)) = T_1(q)$ for $\forall q \in Q_1, \forall A \in \Lambda_1$.
   (d) $\delta_2((q_0, \epsilon, A), (q_0, \epsilon)) = \mathbf{0}$ for $\forall A \in \Lambda_2$.
   (e) $\delta_2 = \mathbf{1}$ for other cases.
(4) $T_2 = \mathbf{0}$.

Let $P_1 = (c_1, \ldots, c_n) \in Path(M_1)$, where $c_i = (p_i, w_i, \gamma_i)$ such that $b(P_1) = c_1 = (q_1, s, Z_1)$ and $e(P_1) = c_n = (q, \epsilon, \gamma)$. Define $c_i' = (p_i, w_i, \gamma_i Z_2)$, for any $i = 1, \ldots, n$ and $c_0' = (q_2, s, Z_2)$. It is easy to see that $(c_0', c_1', \ldots, c_n') \in Path(M_2)$, $|c_0' \vdash_{M_2} c_1'| = \delta_2((q_2, \epsilon, Z_2), (q_1, Z_1Z_2)) = \mathbf{0}$, $|c_i' \vdash_{M_2} c_{i+1}'| = |(p_i, \omega_i, \gamma_i Z_2) \vdash_{M_2} (p_{i+1}, \omega_{i+1}, \gamma_{i+1} Z_2)| = |(p_i, \omega_i, \gamma_i) \vdash_{M_2} (p_{i+1}, \omega_{i+1}, \gamma_{i+1})| = |c_i \vdash_{M_1} c_{i+1}|$ for $i = 1, \ldots, n-1$. Let $c_{n+1}' = (q_0, \epsilon, \gamma Z_2)$, then $(c_0', \ldots, c_n', c_{n+1}') \in Path(M_2)$ and $|c_n' \vdash_{M_2} c_{n+1}'| = |(q_1, \epsilon, \gamma Z_2) \vdash_{M_2} (q_0, \epsilon, \gamma Z_2)| = \delta((q_1, \epsilon, \gamma), (q_0, \gamma)) = T_1(q)$.

In the following, we will clear the stack.

If $\gamma = \epsilon$, define $c_{n+2}' = (q_0, \epsilon, \epsilon)$, then $(c_0', \ldots, c_{n+1}', c_{n+2}') \in Path(M_2)$ and $|c_{n+1}' \vdash_{M_2} c_{n+2}'| = \mathbf{0}$. It follows that $|(c_0', \ldots, c_{n+1}', c_{n+2}')| = |c_0' \vdash_{M_2} c_1'| \boxplus \sum_{i=1}^{n-1} |c_i' \vdash_{M_2} c_{i+1}'| \boxplus |c_{n+1}' \vdash_{M_2} c_{n+2}'| = |P_1| \boxplus T_1(q)$.

If $\gamma = A_1 \cdots A_k (A_i \in \Lambda_1)$ and $\gamma \neq \epsilon$. Define $c_{n+i+1}' = (q_0, \epsilon, A_{k-i} \cdots A_k Z_2)$ $(i = 1, \ldots, k)$ and $c_{n+2+k}' = (q_0, \epsilon, \epsilon)$. Then $|c_{n+1+i}' \vdash_{M_2} c_{n+2+i}'| = \mathbf{0}$ $(i = 1, \ldots, k)$. Still $(c_0', \ldots, c_{n+2+k}') \in Path(M_2)$ and $|(c_0', \ldots, c_{n+2+k}')| = |c_0' \vdash_{M_2} c_1'| \boxplus \sum_{i=1}^{n-1} |c_i' \vdash_{M_2} c_{i+1}'| \boxplus |c_n' \vdash_{M_2} c_{n+1}'| \boxplus |c_{n+1}' \vdash_{M_2} c_{n+2}'| \boxplus \sum_{i=1}^{k} |c_{n+1+i}' \vdash_{M_2} c_{n+2+i}'| = |P_1| \boxplus T_1(q)$. From the above discussions, we conclude that for any $s \in \Sigma^*$, $\{|P| \boxplus T(q) : P \in Path(M), b(P) = (q_0, s, Z_0), e(P) = (q, \epsilon, \gamma)\} \subseteq \{|P| : P \in Path(M), b(P) = (q_0, s, Z_0), e(P) = (q, \epsilon, \epsilon)\}$, thus $|M_1|_d^f \geq |M_2|_d^e$.

On the other hand, let $P_2 \in Path(M_2) = (c_0, \ldots, c_m)$ such that $c_0 = (q_2, s, Z_2)$, $c_k = (q_k, \omega_k, \gamma_k)$, $i = 1, \ldots, m-1$, $c_m = (q, \epsilon, \epsilon)$. By the definition of δ_2, then $c_1 = (q_1, s, Z_1Z_2)$ and $r_{c_0 \vdash_{M_2} c_1} = ((q_2, s, Z_2), (q_1, Z_1Z_2))$, otherwise $|c_0 \vdash_{M_2} c_1| = \mathbf{1}$. Assume c_k $(k \geq 2)$ is the first configuration after c_1 such that $q_k \notin Q_1$. Then $q_1, \ldots, q_{k-1} \in Q_1$ and $r_{c_i \vdash_{M_2} c_{i+1}} \in \Delta_1$ $(i = 1, \ldots, k-2)$. Since $r_{c_i \vdash_{M_1} c_{i+1}} \in \Delta_1$, then transition does not involve the bottom stack symbol Z_2. For $c_j = (q_j, w_j, \gamma_j) \in \Gamma_2$, $(j = 2, \ldots, k-1)$, write $\gamma_j = \gamma_j' Z_2$. Let $c_1' = (q_1, s, Z_1)$, $c_j' = (q_j, w_j, \gamma_j') \in \Gamma_1$ $(j = 2, \ldots, k-1)$. We get that $(c_1', \ldots, c_{k-1}') \in Path(M_1)$ and $r_{c_i' \vdash_{M_1} c_{i+1}'} = r_{c_i \vdash_{M_2} c_{i+1}}$ $(i = 1, \ldots, k-1)$. Consider configuration c_k, if $q_k = q_2$, then $|c_{k-1} \vdash_{M_2} c_k| = \mathbf{1}$, so we only need to consider $q_k = q_0$. Denote $c_k = (q_0, w_k, \gamma_k) \in \Gamma_2$. Then $r_{c_{k-1} \vdash_{M_2} c_k} = ((q_{k-1}, \epsilon, A), (q_0, A))$ for some $A \in \Lambda_1 \cup \{\epsilon\}$, otherwise $|c_{k-1} \vdash_{M_2} c_k| = \mathbf{1}$. It indicates that $w_{k-1} = w_k$, $\gamma_{k-1} = \gamma_k$. It is easy to see that $r_{c_j \vdash_{M_2} c_{j+1}} = ((q_0, \epsilon, A), (q_0, \epsilon))$ for some $A_j \in \Lambda_2$ $(j = k, \ldots, m-1)$, otherwise $|P_2| = \mathbf{1}$. So $|c_j \vdash_{M_2} c_{j+1}| = \mathbf{0}$ $(j = k, \ldots, m-1)$. We could denote $c_j = (q_0, w_k, \gamma_j)$ $(j = k, \ldots, m)$. By the selection of P_2, we know that $w_k = \epsilon$. As a result $|P_2| = |c_0 \vdash_{M_2} c_1| \boxplus |(c_1, \ldots, c_{k-1})| \boxplus |c_{k-1} \vdash_{M_2} c_k| \boxplus \sum_{j=k}^{m-1} |c_j \vdash_{M_2} c_{j+1}| = |(c_1', \ldots, c_{k-1}')| \boxplus T_1(q_{k-1})$, where $(c_1', \ldots, c_{k-1}') \in Path(M_1)$, $c_1' = (q_1, \epsilon, Z_1)$, $c_{k-1}' = (q_{k-1}, \epsilon, \gamma_{k-1})$. Thus $|M|_d^e(s) \geq |M|_d^f(s)$.

The second part. For any $\mathcal{E}$PDF M_1, we will construct a M_2 such that for any $s \in \Sigma^*$, $|M_1|_d^e(s) = |M_2|_d^f(s)$. Assume $M_1 = (Q_1, \Sigma, \Lambda_1, \delta_1, q_1, Z_1, T_1)$, define $M_2 = (Q_2, \Sigma, \Lambda_2, \delta_2, q_2, Z_2, T_2)$ as:

(1) $Q_2 = Q_1 \cup \{q_0, q_2\}$ where $q_0, q_2 \notin Q_1$.
(2) $\Lambda_2 = \Lambda_1 \cup \{Z_2\}$ where $Z_2 \notin \Lambda_1$.
(3) for some $\sigma \in \Sigma$, define, δ_2:
  (a) $\delta_2((q_2, \epsilon, Z_2), (q_1, Z_1Z_2)) = \mathbf{0}$.
  (b) $\delta_2|_{\Lambda_1} = \delta_1$.
  (c) $\delta_2((q, \epsilon, Z_2), (q_0, Z_2)) = \mathbf{0}$ for $\forall q \in Q_1$.
  (d) $\delta_2((q_0, \sigma, Z_2), (q_0, Z_2)) = \mathbf{1}$ for $\forall \sigma \in \Sigma$.
  (e) $\delta_2 = \mathbf{1}$ for other cases.
(4) $T_2 = \mathbf{0}$.

Suppose $P_1 = (c_1, \ldots, c_n) \in Path(M_1)$, $b(P_1) = (q_1, s, Z_1)$, $e(P_1) = (q, \epsilon, \epsilon)$. Denote $c_0' = (q_2, s, Z_2)$, $c_i' = (p_i, w_i, \gamma_i Z_2)$ if $c_i = (p_i, w_i, \gamma_i)$ $(i = 1, \ldots, n)$, $c_{n+1}' = (q_0, \epsilon, Z_2)$. Note that $|c_0' \vdash_{M_2} c_1'| = \delta_2((q_2, \epsilon, Z_2), (q_1, Z_1Z_2)) = \mathbf{0}$, $|c_i' \vdash_{M_2} c_{i+1}'| = |c_i \vdash_{M_1} c_{i+1}$ $(i = 1, \ldots, n-1)$, $|c_n' \vdash_{M_2} c_{n+1}'| = \mathbf{0}$. Denote $(c_0', \ldots, c_{n+1}') = P_1' \in Path(M_2)$, then $|P_1| = |P_1'| \boxplus T_2(q_0)$ and $b(P_1') = (q_2, s, Z_2)$, $e(P_1') = (q_0, \epsilon, Z_2)$. Thus $|M_1|_d^e(s) \geq |M_2|_d^f(s)$.

On the other hand, suppose $P_2 = (c_0, \ldots, c_m) \in Path(M_2)$ such that $c_0 = (q_2, s, Z_2)$, $c_k = (q_k, \omega_k, \gamma_k)$, $i = 1, \ldots, m-1$ and $c_m = (q, \epsilon, \gamma)$. By the definition of δ_2, $c_1 = (q_1, s, Z_1Z_2)$ and $r_{c_0 \vdash_{M_2} c_1} = ((q_2, s, Z_2), (q_1, Z_1Z_2))$. Otherwise $|c_0 \vdash_{M_2} c_1| = \mathbf{1}$. Suppose c_k $(k \geq 2)$ is the first configuration after c_1 such that $q_k \notin Q_1$. By this assumption, $q_1, \ldots, q_{k-1} \in Q_1$ and $r_{c_i \vdash_{M_2} c_{i+1}} \in \Lambda_1$ $(i = 1, \ldots, k-2)$. Denote $c_j = (p_j, w_j, \gamma_j) \in \Gamma_2$ $(j = 2, \ldots, k-1)$. These transitions does not involve Z_2 since $r_{c_i \vdash_{M_2} c_{i+1}} \in \Lambda_1$ $(i = 1, \ldots, k-2)$. We could write $c_1' = (q_1, s, Z_1)$, $\gamma_j = \gamma_j' Z_2$ and $c_j' = (p_j, w_j, \gamma_j') \in \Gamma_1$ $(j = 2, \ldots, k-1)$. In fact $(c_1', \ldots, c_{k-1}') \in Path(M_1)$ and $r_{c_i' \vdash_{M_1} c_{i+1}'} = r_{c_i \vdash_{M_2} c_{i+1}}$ $(i = 1, \ldots, k-2)$. If $q_k = q_2$, then $|c_{k-1} \vdash_{M_2} c_k| = \mathbf{1}$, so we just need to consider $q_k = q_0$. Denote $c_k = (q_0, w_k, \gamma_k) \in \Gamma_2$. We get that $r_{c_{k-1} \vdash_{M_2} c_k} = ((p_{k-1}, \epsilon, Z_2), (q_0, Z_2))$, otherwise $|c_{k-1} \vdash_{M_2} c_k| = \mathbf{1}$. It follows that $\gamma_{k-1} = Z_2$, $\gamma_{k-1}' = \epsilon$ and $w_{k-1} = w_k$. If $w_k \neq \epsilon$, then $|c_k \vdash_{M_2} c_{k+1}| = \mathbf{1}$, that is $|P_2| = \mathbf{1}$. So we could assume $w_k = \epsilon$, then $c_k = (q_0, \epsilon, Z_2) = c_m$. As a result, $|P_2| \boxplus T_2(q) = |c_0 \vdash_{M_2} c_1| \boxplus |(c_1, \ldots, c_{k-1})| \boxplus |c_{k-1} \vdash_{M_2} c_k| \boxplus T_2(q_0) = |(c_1', \ldots, c_{k-1}')|$, where $(c_1', \ldots, c_{k-1}') \in Path(M_1)$ and $c_1' = (q_1, s, Z_1)$, $c_{k-1}' = (p_{k-1}, \epsilon, \epsilon)$. So $|M_2|_d^f(s) \geq |M_1|_d^e(s)$. $\square$

In the following, we consider the language of $\mathcal{E}$PDA recognized in width-first mode. Let "$\vdash_M^0$" be the identity relation over Γ. The composition of $\vdash_M$ is defined as: $\vdash_M^{n+1} := \vdash_M^n \circ \vdash_M$. Furthermore, define $|c_a \vdash_M^{n+1} c_b| = \wedge_c |c_a \vdash_M^n c| \boxplus |c \vdash_M c_b|$ recursively as

$$|c_a \vdash_M^n c_b| = \bigwedge_{c_{n-1}} \left(\cdots \left(\bigwedge_{c_2} \left(\bigwedge_{c_1} |c_a \vdash_M c_1| \boxplus |c_1 \vdash_M c_2| \right) \boxplus |c_2 \vdash_M c_3| \right) \cdots \right) \boxplus |c_{n-1} \vdash_M c_b|$$

Definition 8.4. The $\mathcal{E}$-valued language accepted by an $\mathcal{E}$PDA $M = (Q, \Sigma, \Lambda, \delta, q_0, Z_0, T)$ by final state in a width-first mode, denoted by $|M|_w^f$, is defined as: for any $s \in \Sigma^*$,

$$|M|_w^f(s) = \bigwedge_{n \geq 0, q \in Q, \gamma \in \Lambda^*} |(q_0, s, Z_0) \vdash_M^n (q, \epsilon, \gamma)| \boxplus T(q) \tag{16}$$

Definition 8.5. The $\mathcal{E}$-valued language accepted by an $\mathcal{E}$PDA $M = (Q, \Sigma, \Lambda, \delta, q_0, Z_0, T)$ by empty stack in a width-first mode, denoted by $|M|_w^e$, is defined as: for any $s \in \Sigma^*$,

$$|M|_w^e(s) = \bigwedge_{n\geq 0, q\in Q} |(q_0, s, Z_0) \vdash_M^n (q, \epsilon, \epsilon)| \tag{17}$$

The following theorem carefully compare recognizability of $\mathcal{E}$ valued pushdown automaton in the depth-first mode with that in the width-first mode.

Theorem 8.2. *The following conditions are equivalent:*

(i) $a \boxplus (b \wedge c) = (a \boxplus b) \wedge (a \boxplus c)$ *for any* $a, b, c \in \mathcal{E}$.
(ii) $|M|_d^f = |M|_w^f$ *for any* $M \in$*PDA*$(\mathcal{E}, \Sigma)$.
(iii) $|M|_d^e = |M|_w^e$ *for any* $M \in$*PDA*$(\mathcal{E}, \Sigma)$.

Proof. (i)⇒(ii), (iii): If $\boxplus$ distributes over $\wedge$, then for any $s \in \Sigma^*$

$$\begin{aligned}
|M|_w^f(s) &= \wedge_{n\geq 0} \wedge_{q\in Q,\gamma\in\Gamma^*} [\wedge_{c_{n-1}}(\cdots(\wedge_{c_1}|(q_0, s, Z_0) \vdash_M c_1| \boxplus |c_1 \vdash_M c_2|)\cdots) \boxplus |c_{n-1} \vdash_M (q, \epsilon, \gamma)|] \boxplus T(q)\\
&= \wedge_{n\geq 0} \wedge_{q\in Q,\gamma\in\Gamma^*} \wedge_{c_i}(|(q_0, s, Z_0) \vdash_M c_1| \boxplus \cdots \boxplus |c_{n-1} \vdash_M (q, \epsilon, \gamma)| \boxplus T(q))\\
&= \wedge_{n\geq 0} \wedge \{|P| \boxplus T(e(P)) : P \in Path_n(M)\}\\
&= \wedge \{|P| \boxplus T(e(P)) : P \in Path(M)\}\\
&= |M|_d^f(s)
\end{aligned}$$

and similarly $|M|_w^e(s) = |M|_d^e(s)$.

(ii)⇒(i): Take arbitrary elements $a, b, c \in \mathcal{E}$. Consider $M = (Q, \Sigma, \Lambda, \delta, q_0, Z_0, T)$ where $Q = \{q_0, q_1, q_2, q_3\}$, $\Lambda = \{Z_0, Z_1, Z_2, Z_3\}$ and $T = \mathbf{c}$. Suppose $\sigma \in \Sigma$, define δ as:

$$\begin{aligned}
&\delta((q_0, \sigma, Z_0), (q_1, Z_1)) = a\\
&\delta((q_0, \sigma, Z_0), (q_2, Z_2)) = b\\
&\delta((q_1, \sigma, Z_1), (q_3, Z_3)) = \mathbf{0}\\
&\delta((q_2, \sigma, Z_2), (q_3, Z_3)) = \mathbf{0}\\
&\delta = \mathbf{1} \quad \text{for other cases}
\end{aligned}$$

then

$$\begin{aligned}
|M|_w^f(\sigma\sigma) &= \wedge_{n\geq 0} \wedge_{q\in Q,\gamma\in\Lambda^*} [\wedge_{c_{n-1}}(\cdots(\wedge_{c_1}|(q_0, s, Z_0) \vdash_M c_1| \boxplus |c_1 \vdash_M c_2|)\cdots) \boxplus |c_{n-1} \vdash_M (q, \epsilon, \gamma)|] \boxplus T(q)\\
&= [(|(q_0, \sigma\sigma, Z_0) \vdash_M (q_1, \sigma, Z_1)| \boxplus |(q_1, \sigma, Z_0) \vdash_M (q_3, \epsilon, Z_3)|)\\
&\quad \wedge (|(q_0, \sigma\sigma, Z_0) \vdash_M (q_2, \sigma, Z_2)| \boxplus |(q_2, \sigma, Z_2) \vdash_M (q_3, \epsilon, Z_3)|)] \boxplus T(q_3)\\
&= (a \wedge b) \boxplus c
\end{aligned}$$

and $|M|_d^f(\sigma\sigma) = (a \boxplus c) \wedge (b \boxplus c)$. So we infer that $(a \boxplus c) \wedge (b \boxplus c) = (a \wedge b) \boxplus c$ for any $a, b, c \in \mathcal{E}$.

(iii)⇒(i): Take arbitrary elements $a, b, c \in \mathcal{E}$. Consider $M = (Q, \Sigma, \Lambda, \delta, q_0, Z_0, T)$ where $Q = \{q_0, q_1, q_2, q_3\}$, $\Lambda = \{Z_0, Z_1\}$. Suppose $\sigma \in \Sigma$ and define δ as:

$$\begin{aligned}
&\delta((q_0, \sigma, Z_0), (q_1, Z_1)) = a\\
&\delta((q_0, \sigma, Z_0), (q_2, Z_1)) = b\\
&\delta((q_1, \sigma, Z_1), (q_3, Z_1)) = \mathbf{0}\\
&\delta((q_2, \sigma, Z_1), (q_3, Z_1)) = \mathbf{0}\\
&\delta((q_3, \sigma, Z_1), (q_3, \epsilon)) = c\\
&\delta = \mathbf{1} \quad \text{for other cases}
\end{aligned}$$

then

$$\begin{aligned}
|M|_w^e(\sigma\sigma\sigma) &= \wedge_{n\geq 0} \wedge_{q\in Q} (\wedge_{c_{n-1}}(\cdots(\wedge_{c_1}|(q_0, \sigma, Z_0) \vdash_M c_1| \boxplus |c_1 \vdash_M c_2|)\cdots) \boxplus |c_{n-1} \vdash_M (q, \epsilon, \epsilon)|)\\
&= [(|(q_0, \sigma\sigma\sigma, Z_0) \vdash_M (q_1, \sigma\sigma, Z_1)| \boxplus |(q_1, \sigma\sigma, Z_1) \vdash_M (q_3, \sigma, Z_1)|)\\
&\quad \wedge (|(q_0, \sigma\sigma\sigma, Z_0) \vdash_M (q_2, \sigma\sigma, Z_1)| \boxplus |(q_2, \sigma\sigma, Z_1) \vdash_M (q_3, \sigma, Z_1)|)] \boxplus |(q_3, \sigma, Z_1) \vdash_M (q_3, \epsilon, \epsilon)|\\
&= (a \wedge b) \boxplus c
\end{aligned}$$

and $|M|_d^e(\sigma\sigma\sigma) = (a \boxplus c) \wedge (b \boxplus c)$. So it is inferred that $(a \boxplus c) \wedge (b \boxplus c) = (a \wedge b) \boxplus c$ for any $a, b, c \in \mathcal{E}$. □

According to Theorems 8.1 and 8.2, we have shown that the language recognizability by final states is also equivalent to the language recognizability by empty stack in the width-first mode provided that $\boxplus$ distribute over $\wedge$ in $\mathcal{E}$.

Corollary 8.3. *Let ε be an MV algebra. For any $\mathcal{E}$PDA M_1, there is an $\mathcal{E}$PDA M_2 such that $|M_1|_w^f = |M_2|_w^e$. Conversely, for any $\mathcal{E}$PDA M_1, there is an $\mathcal{E}$PDA M_2 such that $|M_1|_w^e = |M_2|_w^f$.*

9. Relations between $\mathcal{E}$-valued context free grammars and $\mathcal{E}$-valued pushdown automata

In this section, we will demonstrate the equivalence of $\mathcal{E}$-valued context free grammars and $\mathcal{E}$-valued pushdown automata. Here, by equivalence, we mean that an $\mathcal{E}$-valued context free grammars can be simulated by an $\mathcal{E}$-valued pushdown automaton and conversely that an $\mathcal{E}$-valued pushdown automaton can be simulated by an $\mathcal{E}$-valued context free grammar.

Let $G = (V, \Sigma, S, \mu)$ be an $\mathcal{E}$GNF. With the technique used in classical automata theory, we construct an $M^G = (\{q\}, \Sigma, V, \delta, q, S, T) \in$PDA$(\mathcal{E}, \Sigma)$ as follows: $\delta((q, \sigma, A), (q, \gamma)) = \mu(A \to \sigma\gamma)$ for $\forall A \to \sigma\gamma \in supp(G)$ and $\delta = \mathbf{1}$ for others; $T = \mathbf{0}$.

The following Theorems 9.1 and 9.2 combined with Theorem 7.8 show that for the equivalence between $\mathcal{E}$-valued context free grammar and $\mathcal{E}$-valued PDA, the depth-first mode and distributivity of $\boxplus$ over $\wedge$ is not required.

Theorem 9.1. *Let G be an $\mathcal{E}$GNF, then $|G|_d = |M^G|_d^e$, where M^G is the $\mathcal{E}$PDA constructed above.*

Proof. First, we prove that for $\forall P \in Path_n(M)$ with $b(P) = (q, w, S)$, $e(P) = (q, \epsilon, \alpha)$, there is $\kappa \in D_G^n(S, w\alpha)$ such that $|P| = |\kappa|$, where $w \in \Sigma^*$, $\alpha \in V^*$.

If $n = 1$, $P = ((q, w, S), (q, \epsilon, \alpha))$, then $w \in \Sigma_\epsilon$ and $r_{(q,w,S)\vdash_{M^G}(q,\epsilon,\alpha)} = ((q, w, S), (q, \alpha))$. Thus $\kappa = ((S, w\alpha), (S \to w\alpha)) \in D_G^1(S, w\alpha)$ satisfies $|\kappa| = |P|$.

Assume then that the hypothesis holds for $n \le k$. When $n = k + 1$, for any $P \in Path_{k+1}(M^G)$, $b(P) = (q, w, S)$, $e(P) = (q, \epsilon, \alpha)$, we can write $P = ((q, w, S), \dots, (q, \sigma, \gamma), (q, \epsilon, \alpha))$, where $w = w'\sigma$, $\sigma \in \Sigma_\epsilon$. If denote $\gamma = A\beta$, $\alpha = \gamma'\beta$, then $r((q, \sigma, \gamma) \vdash_{M^G} (q, \epsilon, \alpha)) = ((q, \sigma, A), (q, \gamma'))$. With the same transitions in P, there is $P' = ((q, w', S), \dots, (q, \epsilon, \gamma)) \in Path_k(M^G)$. By the hypothesis we get a $\kappa' \in D_G^k(S, w'\gamma)$ and $|\kappa'| = |P'|$. Define $\kappa = ((S, w'\gamma, w\alpha), (p_1, \dots, p_{k+1})) \in D_G^{k+1}(S, w\alpha)$ if $\kappa' = ((S, \dots, w'\gamma), (p_1, \dots, p_k)$, then $|\kappa| = |\kappa'| \boxplus \mu(A \to \sigma\alpha') = |P'| \boxplus \delta((q, \sigma, A), (q, \alpha')) = |P|$.

Conversely we prove that for $\forall \kappa \in D_G^n(S, w\alpha)$ where $w \in \Sigma^*$, $\alpha \in V^*$, there is $P \in Path_n(M^G)$ such that $b(P) = (q, w, S)$, $e(P) = (q, \epsilon, \alpha)$ and $|\kappa| = |P|$.

If $n = 1$, $\kappa = (S, S \to w\alpha, w\alpha)$ where $w \in \Sigma$. So $P = ((q, w, S), (q, \epsilon, \alpha))$ satisfies $|P| = |\kappa|$.

Assume that the hypothesis holds for $n = k$. When $n = k + 1$, suppose $\kappa = ((S, \dots, w'\gamma, w\alpha), (p_1, \dots, p_{k+1}))$, $w = w'\sigma$, $\gamma = A\beta$, $\alpha = \alpha'\beta$ and $p_{k+1} = A \to \sigma\alpha'$. Take $\kappa' = ((S, \dots, w'\gamma), (p_1, \dots, p_k)) \in D_G^k(S, w'\gamma)$ as the front part of κ. By the hypothesis, there is $P' \in Path_k(M)$, $b(P') = (q, w', S)$, $e(P') = (q, \epsilon, \gamma)$ and $|P'| = |\kappa'|$. With the same transitions, there is $P'' \in Path_k(M^G)$, $b(P'') = (q, w, S)$, $e(P'') = (q, \sigma, \gamma)$ and $|P''| = |P'|$. Let $P = P''((q, \sigma, \gamma), (q, \epsilon, \alpha))$, that is concatenation of P'' and $((q, \sigma, \gamma), (q, \epsilon, \alpha))$, then $|P| = |P''| \boxplus \delta((q, \sigma, A), (q, \alpha')) = |\kappa'| \boxplus \mu(A \to \sigma\alpha') = |\kappa|$.

As a result, for $\forall \kappa \in D_G^+(S, s)$ there is $P \in Path(M^G)$ such that $b(P) = (q, s, S)$, $e(P) = (q, \epsilon, \epsilon)$, $|\kappa| = |P|$, and vise versa. So $|G|_d = |M^G|_d^e$. □

Let $M = (Q, \Sigma, \Lambda, \delta, q_0, Z_0, T) \in$PDA$(\mathcal{E}, \Sigma)$. We will construct an equivalent $\mathcal{E}$-valued grammar using the same technique with classical automata theory. Construct an $\mathcal{E}$CFG $G^M = (V, \Sigma, S, \mu)$ as follows:

(1) $V = \{S\} \cup \{[p, A, q] : p, q \in Q, A \in \Lambda\}$,
(2) $\mu([q, A, q_{n+1}] \to \sigma[q_1, A_1, q_2][q_2, A_2, q_3] \cdots [q_n, A_n, q_{n+1}]) = \delta((q, \sigma, A), (q_1, A_1 \cdots A_n))$ for $\forall q, q_1, \dots, q_n, q_{n+1} \in Q$, $\sigma \in \Sigma_\epsilon, A, A_1, \dots, A_n \in \Lambda$, where $n \ge 1$,
(3) $\mu([p, A, q] \to \sigma) = \delta((p, \sigma, A), (q, \epsilon))$ for $\forall p, q \in Q$,
(4) $\mu(S \to [q_0, Z_0, q]) = \mathbf{0}$ for $\forall q \in Q$,
(5) $\mu = \mathbf{1}$ for other cases.

Theorem 9.2. *Let $M \in$PDA$(\mathcal{E}, \Sigma)$, then $|G^M|_d = |M|_d^e$ where G^M is the $\mathcal{E}$CFG constructed as above.*

Proof. Let $\kappa \in D_{G^M}^n([q, A, q'], w)$, $q, q' \in Q$, $A \in \Lambda$, $w \in \Sigma^*$. We show recursively that there is $P \in Path(M)$ satisfying $b(P) = (q, w, A)$, $e(P) = (q', \epsilon, \epsilon)$ and $|\kappa| = |P|$.

If $n = 1$, $\kappa = (([q, A, q'], w), p_1)$, it follows that $w \in \Sigma_\epsilon$ or $|\kappa| = \mathbf{1}$. So $|\kappa| = \mu([q, A, q'] \to w) = \delta((q, w, A), (q', \epsilon)) = |(q_1, w, A) \vdash (q', \epsilon, \epsilon)| = |P|$ where $P = ((q, w, A), (q', \epsilon, \epsilon))$.

Assume the hypothesis holds for $n \le k$. When $n = k + 1$, let $\kappa = (([q, A, q'], \alpha_1, \dots, \omega), (p_1, \dots, p_{k+1})) \in D_{G^M}^{k+1}([q, A, q'], w)$, then there are $\alpha_1 = \sigma[q_1, A_1, q_2] \cdots [q_m, A_m, q']$ for some $q_i \in Q$, $A_i \in \Lambda$, $\sigma \in \Sigma_\epsilon$ and $\mu(p_1) = \delta((q, \sigma, A), (q_1, A_1 \cdots A_m))$. Suppose $\kappa_i \in D_{G^M}^{n_i}([q_i, A_i, q_{i+1}], w_i)$, $(i = 1, \dots, m)$, $q_{m+1} = q'$, where $\sum n_i = k$, $w = \sigma w_1 \cdots w_m$. By the hypothesis there is $P_i \in Path(M)$ such that $b(P_i) = (q_i, w_i, A_i)$, $e(P_i) = (q_{i+1}, \epsilon, \epsilon)$ and $|\kappa_i| = |P_i|$. With the same transitions in P_i we can find $P_i' \in Path(M)$ satisfying

(1) $b(P_i') = (q_i, w_i \cdots w_m, A_i \cdots A_m)$ $(i = 1, \dots, m)$,
(2) $e(P_j') = b(P_{j+1}')$ $(j = 1, \dots, m - 1)$, $e(P_m') = (q', \epsilon, \epsilon)$,
(3) $|P_i'| = |P_i|$.

Denote $P = ((q, w, A), (q_1, w_1 \cdots w_m, A_1 \cdots A_m))P'_1 \cdots P'_m$, then $|P| = \delta((q, \sigma, A), (q_1, A_1 \cdots A_m)) \boxplus \sum_{i=1}^m |P_i| = \mu(p_1) \boxplus \sum_{i=1}^m |\kappa_i| = |\kappa|$ and $b(P) = (q, w, A), e(P) = (q', \epsilon, \epsilon)$.

On the other hand we show that for any $P \in Path_n(M)$ in which $b(P) = (q, w, A), e(P) = (q', \epsilon, \epsilon)$, there is $\kappa \in D^+_{G^M}([q, A, q'], w)$ such that $|P| = |\kappa|$.

If $n = 1$, $P = ((q, w, A), (q', \epsilon, \epsilon))$, which indicates that $w \in \Sigma_\epsilon$. Thus $|P| = \delta((q, w, A), (q', \epsilon)) = \mu([q, A, q'] \to w) = |\kappa|$ where $\kappa = ([q, A, q'], p_1, w) \in D^1_{G^M}([q, A, q'], w)$.

Assume the hypothesis holds for $n \le k$. When $n = k + 1$, let $P = (c_0, \ldots, c_k, c_{k+1}) \in Path_{k+1}(M)$, where $c_0 = (q, w, A), c_{k+1} = (q', \epsilon, \epsilon)$. Suppose $w = \sigma w'$, $c_1 = (q_1, w', A_1 \cdots A_m)$, then $|c_0 \vdash c_1| = \delta((q, \sigma, A), (q_1, A_1 \cdots A_m))$. We could divide $w' = w_1 \cdots w_m$ satisfying that A_i is the top symbol in the stack when $w_i \cdots w_m$ left. Assume $c_{k_i} = (q_i, w_i \cdots w_m, A_i \cdots A_m)$, $1 \le k_i \le k$. There are $P_i \in Path(M)$ such that $|P_i| = |(c_{k_i}, \ldots, c_{k_{i+1}})|$ and $b(P_i) = (q_i, w_i, A_i), e(P_i) = (q_{i+1}, \epsilon, \epsilon)$. By hypothesis, there are $\kappa_i \in D^+_{G^M}([q_i, A_i, q_{i+1}], w_i)$, $|\kappa_i| = |P_i|$. So there is $\kappa = ([q, A, q_{m+1}], p_1, \sigma[q_1, A_1, q_2] \cdots [q_m, A_m, q_{m+1}], \ldots, \sigma w_1 \cdots w_m = w)$ with the same transitions of κ_i. Thus $|\kappa| = \mu(p_1) \boxplus \sum |\kappa_i| = \delta((q_1, \sigma, A), (q_1, A_1 \cdots A_m)) \boxplus \sum |P_i| = |P|$.

We get that for $\forall \kappa \in D^+_{G^M}([q_0, Z_0, q'], s)$ there is some $P \in Path(M)$ such that $b(P) = (q, s, Z_0), e(P) = (q', \epsilon, \epsilon)$, and vise versa. Since there is a one-to-one corresponding relationship between $D^+_{G^M}(S, s)$ and $\bigcup\{D^+_{G^M}([q_0, Z_0, q], s) : q \in Q\}$, so $|G^M|_d = |M|^e_d$ from Eqs. (11) and (15). $\square$

However, when using the width-first principle, from the following Theorems 9.3 and 9.5, we will find that the equivalence between $\mathcal{E}$-valued context free grammar and $\mathcal{E}$ valued PDA requires the distributivity of $\boxplus$ over $\wedge$ in $\mathcal{E}$. That is, $\mathcal{E}$ is an MV algebra.

Theorem 9.3. *$|G|_w = |M^G|^e_w$ for any $G \in$GNF$(\mathcal{E}, \Sigma)$ iff $a \boxplus (b \wedge c) = (a \boxplus b) \wedge (a \boxplus c)$ for any $a, b, c \in \mathcal{E}$.*

Proof. "If part": Conclude from Theorems 7.1, 8.2 and 9.1.

"Only if part": For any $a, b, c \in \mathcal{E}$ construct a $G = (V, \Sigma, S, \mu) \in$GNF$(\mathcal{E}, \Sigma)$ as follows: $V = \{S, A, B, C\}$, for some $\sigma \in \Sigma$. μ is defined as:

$$\mu(S \to \sigma AB) = \mathbf{0}, \mu(S \to \sigma AC) = \mathbf{0}$$
$$\mu(A \to \sigma) = a, \mu(B \to \sigma) = b, \mu(C \to \sigma) = c$$

and $\mu = \mathbf{1}$ for the rest.

Take $s = \sigma\sigma\sigma$,

$$\begin{aligned}
|G|_w(s) &= \bigwedge_{\gamma_2} \left(\bigwedge_{\gamma_1} \widetilde{\mu}(S \Rightarrow \gamma_1) \boxplus \widetilde{\mu}(\gamma_1 \Rightarrow \gamma_2) \right) \boxplus \widetilde{\mu}(\gamma_2 \Rightarrow s) \\
&= \left[\left(\bigwedge_{\gamma_1} \widetilde{\mu}(S \Rightarrow \gamma_1) \boxplus \widetilde{\mu}(\gamma_1 \Rightarrow \sigma\sigma B) \right) \boxplus \widetilde{\mu}(\sigma\sigma B \Rightarrow s) \right] \\
&\quad \bigwedge \left[\left(\bigwedge_{\gamma_1} \widetilde{\mu}(S \Rightarrow \gamma_1) \boxplus \widetilde{\mu}(\gamma_1 \Rightarrow \sigma\sigma C) \right) \boxplus \widetilde{\mu}(\sigma\sigma C \Rightarrow s) \right] \\
&\quad \bigwedge \left[\left(\bigwedge_{\gamma_1} \widetilde{\mu}(S \Rightarrow \gamma_1) \boxplus \widetilde{\mu}(\gamma_1 \Rightarrow \sigma A\sigma) \right) \boxplus \widetilde{\mu}(\sigma A\sigma \Rightarrow s) \right] \\
&= [\widetilde{\mu}(S \Rightarrow \sigma AB) \boxplus \widetilde{\mu}(\sigma AB \Rightarrow \sigma\sigma B) \boxplus \widetilde{\mu}(\sigma\sigma B \Rightarrow s)] \wedge [\widetilde{\mu}(S \Rightarrow \sigma AC) \boxplus \widetilde{\mu}(\sigma AC \Rightarrow \sigma\sigma B) \boxplus \widetilde{\mu}(\sigma\sigma C \Rightarrow s)] \\
&\quad \wedge [((\widetilde{\mu}(S \Rightarrow \sigma AB) \boxplus \widetilde{\mu}(\sigma AB \Rightarrow \sigma A\sigma)) \wedge (\widetilde{\mu}(S \Rightarrow \sigma AC) \boxplus \widetilde{\mu}(\sigma AC \Rightarrow \sigma A\sigma))) \boxplus \widetilde{\mu}(\sigma A\sigma \Rightarrow s)] \\
&= [\mu(S \to \sigma AB) \boxplus \mu(A \to \sigma) \boxplus \mu(B \to \sigma)] \wedge [\mu(S \to \sigma AC) \boxplus \mu(A \to \sigma) \boxplus \mu(C \to \sigma)] \\
&\quad \wedge [((\mu(S \to \sigma AB) \boxplus \mu(B \to \sigma)) \wedge (\mu(S \to \sigma AC) \boxplus \mu(C \to \sigma))) \boxplus \mu(A \to \sigma)] \\
&= (b \wedge c) \boxplus a
\end{aligned}$$

The corresponding $M^G = (\{q\}, \Sigma, V, \delta, q, S, T)$ where

$$\begin{aligned}
\delta((q, \sigma, S), (q, AB)) &= \mu(S \to \sigma AB) = \mathbf{0} \\
\delta((q, \sigma, S), (q, AC)) &= \mu(S \to \sigma AC) = \mathbf{0} \\
\delta((q, \sigma, A), (q, \epsilon)) &= \mu(A \to \sigma) = a \\
\delta((q, \sigma, B), (q, \epsilon)) &= \mu(B \to \sigma) = b \\
\delta((q, \sigma, C), (q, \epsilon)) &= \mu(C \to \sigma) = c
\end{aligned}$$

and $\delta = \mathbf{1}$ for the rest,

$$
\begin{aligned}
|M^G|_w^e(s) &= \bigwedge_{c_2} \left(\bigwedge_{c_1} |(q, s, S) \vdash c_1| \boxplus |c_1 \vdash c_2| \right) \boxplus |c_2 \vdash (q, \epsilon, \epsilon)| \\
&= [|(q, s, S) \vdash (q, \sigma\sigma, AB)| \boxplus |(q, \sigma\sigma, AB) \vdash (q, \sigma, B)| \boxplus |(q, \sigma, B) \vdash (q, \epsilon, \epsilon)|] \\
&\quad \wedge [|(q, s, S) \vdash (q, \sigma\sigma, AC)| \boxplus |(q, \sigma\sigma, AC) \vdash (q, \sigma, C)| \boxplus |(q, \sigma, C) \vdash (q, \epsilon, \epsilon)|] \\
&= [\delta((q, \sigma, S), (q, AB)) \boxplus \delta((q, \sigma, A), (q, \epsilon)) \boxplus \delta((q, \sigma, B), (q, \epsilon))] \\
&\quad \wedge [\delta((q, \sigma, S), (q, Ac)) \boxplus \delta((q, \sigma, A), (q, \epsilon)) \boxplus \delta((q, \sigma, C), (q, \epsilon))] \\
&= (a \boxplus b) \wedge (a \boxplus c)
\end{aligned}
$$

thus $|G|_w = |M^G|_w^e$ implies $(a \boxplus b) \wedge (a \boxplus c) = a \boxplus (b \wedge c)$. □

From the Theorems 8.1 and 9.2 we get the following corollary:

Corollary 9.4. *Let $M \in PDA(\mathcal{E}, \Sigma)$, there exists an $\mathcal{E}$CFG G such that $|G|_d = |M|_d^f$.*

Theorem 9.5. *$|G^M|_w = |M|_w^e$ for any $M \in PDA(\mathcal{E}, \Sigma)$ iff $a \boxplus (b \wedge c) = (a \boxplus b) \wedge (a \boxplus c)$ for any $a, b, c \in \mathcal{E}$.*

Proof. "If part". Follows easily from Theorems 7.1, 8.2 and 9.2.

"Only if part". For any given $a, b, c \in \mathcal{E}$, construct $M = (Q, \Sigma, \Lambda, \delta, q_0, Z_0, T)$ as follows:

(1) $Q = \{q_0, q_1, q_2, q_3, q_4, q_5, q_6\}$.
(2) assume $\sigma \in \Sigma$.
(3) $\Lambda = \{Z_0, Z_1, Z_2, Z_3\}$.
(4) δ is defined as:

$$
\begin{aligned}
&\delta((q_0, \epsilon, Z_0), (q_2, Z_3Z_1Z_1)) = \mathbf{0} \\
&\delta((q_0, \epsilon, Z_0), (q_2, Z_3Z_2Z_2)) = \mathbf{0} \\
&\delta((q_2, \sigma, Z_3), (q_4, \epsilon)) = a \\
&\delta((q_4, \sigma, Z_1), (q_5, \epsilon)) = b \\
&\delta((q_6, \sigma, Z_2), (q_1, \epsilon)) = c \\
&\delta((q_5, \sigma, Z_1), (q_1, \epsilon)) = \mathbf{0} \\
&\delta((q_4, \sigma, Z_2), (q_6, \epsilon)) = \mathbf{0}
\end{aligned}
$$

and $\delta = \mathbf{1}$ for other cases.

(5) $T = \mathbf{0}$.

Let $s = \sigma\sigma\sigma$, the $\mathcal{E}$-value of s accepted by M is

$$
\begin{aligned}
|M|_w^e(s) &= \wedge_{n\geq 0} \wedge_{q\in Q} (\wedge_{c_{n-1}}(\cdots(\wedge_{c_1}|(q_0, s, Z_0) \vdash_M c_1| \boxplus |c_1 \vdash_M c_2|)\cdots) \boxplus |c_{n-1} \vdash_M (q, \epsilon, \epsilon)|) \\
&= \wedge_{n\geq 0} \wedge_{c_{n-1}} (\cdots(\wedge_{c_1}|(q_0, s, Z_0) \vdash_M c_1| \boxplus |c_1 \vdash_M c_2|)\cdots) \boxplus |c_{n-1} \vdash_M (q_1, \epsilon, \epsilon)| \\
&= \wedge_{n\geq 0}[\wedge_{c_{n-2}}(\cdots(\wedge_{c_1}|(q_0, s, Z_0) \vdash_M c_1| \boxplus |c_1 \vdash_M c_2|)\cdots) \boxplus |(q_5, \sigma, Z_1) \vdash_M (q_1, \epsilon, \epsilon)|] \\
&\quad \wedge [\wedge_{c_{n-2}}(\cdots(\wedge_{c_1}|(q_0, s, Z_0) \vdash_M c_1| \boxplus |c_1 \vdash_M c_2|)\cdots) \boxplus |(q_6, \sigma, Z_2) \vdash_M (q_1, \epsilon, \epsilon)|] \\
&= [|(q_0, s, Z_0) \vdash_M (q_2, s, Z_3Z_1Z_1)| \boxplus |(q_2, s, Z_3Z_1Z_1) \vdash_M (q_4, \sigma\sigma, Z_1Z_1)| \\
&\quad \boxplus |(q_4, \sigma\sigma, Z_1Z_1) \vdash_M (q_5, \sigma, Z_1)| \boxplus |(q_5, \sigma, Z_1) \vdash_M (q_1, \epsilon, \epsilon)] \\
&\quad \wedge [|(q_0, s, Z_0) \vdash_M (q_2, s, Z_3Z_2Z_2)| \boxplus |(q_2, s, Z_3Z_2Z_2) \vdash_M (q_4, \sigma\sigma, Z_2Z_2)| \\
&\quad \boxplus |(q_4, \sigma\sigma, Z_2Z_2) \vdash_M (q_6, \sigma, Z_2)| \boxplus |(q_6, \sigma, Z_2) \vdash_M (q_1, \epsilon, \epsilon)] \\
&= [\delta((q_0, \epsilon, Z_0), (q_2, Z_3Z_1Z_1)) \boxplus \delta((q_2, \sigma, Z_3), (q_4, \epsilon)) \boxplus \delta((q_4, \sigma, Z_1), (q_5, \epsilon)) \boxplus \delta((q_5, \sigma, Z_1), (q_1, \epsilon))] \\
&\quad \wedge [\delta((q_0, \epsilon, Z_0), (q_2, Z_3Z_2Z_2)) \boxplus \delta((q_2, \sigma, Z_3), (q_4, \epsilon)) \boxplus \delta((q_4, \sigma, Z_2), (q_6, \epsilon)) \boxplus \delta((q_6, \sigma, Z_2), (q_1, \epsilon))] \\
&= [\mathbf{0} \boxplus a \boxplus b \boxplus \mathbf{0}] \wedge [\mathbf{0} \boxplus a \boxplus \mathbf{0} \boxplus c] \\
&= (a \boxplus b) \wedge (a \boxplus c)
\end{aligned}
$$

The corresponding $G^M = (V, \Sigma, S, \mu)$ is:

(1) $V = \{S\} \cup \{[p, Z, q] : p, q \in Q, Z \in \Lambda\}$.
(2) μ is constructed as:

$$\mu([q_0, Z_0, q_1] \to [q_2, Z_3, q_i][q_i, Z_1, q_j][q_j, Z_1, q_1]) = \delta((q_0, \epsilon, Z_0), (q_2, Z_3Z_1Z_1)) = \mathbf{0} \quad \text{for } \forall i, j$$
$$\mu([q_0, Z_0, q_1] \to [q_2, Z_3, q_i][q_i, Z_2, q_j][q_j, Z_2, q_1]) = \delta((q_0, \epsilon, Z_0), (q_2, Z_3Z_2Z_2)) = \mathbf{0} \quad \text{for } \forall i, j$$
$$\mu([q_2, Z_3, q_4] \to \sigma) = \delta((q_2, \sigma, Z_3), (q_4, \epsilon)) = a$$
$$\mu([q_4, Z_1, q_5] \to \sigma) = \delta((q_4, \sigma, Z_1), (q_5, \epsilon)) = b$$
$$\mu([q_6, Z_2, q_1] \to \sigma) = \delta((q_6, \sigma, Z_2), (q_1, \epsilon)) = c$$
$$\mu([q_5, Z_1, q_1] \to \sigma) = \delta((q_5, \sigma, Z_1), (q_1, \epsilon)) = \mathbf{0}$$
$$\mu([q_4, Z_2, q_6] \to \sigma) = \delta((q_4, \sigma, Z_2), (q_6, \epsilon)) = \mathbf{0}$$
$$\mu(S \to [q_0, Z_0, q_i]) = \mathbf{0} \quad \text{for } \forall i$$

The $\mathcal{E}$-value of s accepted by G^M is

$$\begin{aligned}
|G^M|_w(s) &= \bigwedge_{m\geq 1}\left(\bigwedge_{\gamma_{m-1}}\left(\cdots\left(\bigwedge_{\gamma_1}\widetilde{\mu}(S \Rightarrow \gamma_1) \boxplus \widetilde{\mu}(\gamma_1 \Rightarrow \gamma_2)\right)\cdots\right) \boxplus \widetilde{\mu}(\gamma_{m-1} \Rightarrow s)\right) \\
&= \bigwedge_{\gamma_4}\left(\bigwedge_{\gamma_3}\left(\bigwedge_{\gamma_2}\widetilde{\mu}([q_0, Z_0, q_1] \Rightarrow \gamma_2) \boxplus \widetilde{\mu}(\gamma_2 \Rightarrow \gamma_3)\right) \boxplus \widetilde{\mu}(\gamma_3 \Rightarrow \gamma_4)\right) \boxplus \widetilde{\mu}(\gamma_4 \Rightarrow s) \\
&= \left[\left(\bigwedge_{\gamma_3}\left(\bigwedge_{\gamma_2}\widetilde{\mu}([q_0, Z_0, q_1] \Rightarrow \gamma_2) \boxplus \widetilde{\mu}(\gamma_2 \Rightarrow \gamma_3)\right) \boxplus \widetilde{\mu}(\gamma_3 \Rightarrow [q_2, Z_3, q_4]\sigma\sigma)\right) \boxplus \widetilde{\mu}([q_2, Z_3, q_4]\sigma\sigma \Rightarrow s)\right] \\
&\quad \bigwedge \left[\left(\bigwedge_{\gamma_3}\left(\bigwedge_{\gamma_2}\widetilde{\mu}([q_0, Z_0, q_1] \Rightarrow \gamma_2) \boxplus \widetilde{\mu}(\gamma_2 \Rightarrow \gamma_3)\right) \boxplus \widetilde{\mu}(\gamma_3 \Rightarrow \sigma[q_4, Z_1, q_5]\sigma)\right) \boxplus \widetilde{\mu}(\sigma[q_4, Z_1, q_5]\sigma \Rightarrow s)\right] \\
&\quad \bigwedge \left[\left(\bigwedge_{\gamma_3}\left(\bigwedge_{\gamma_2}\widetilde{\mu}([q_0, Z_0, q_1] \Rightarrow \gamma_2) \boxplus \widetilde{\mu}(\gamma_2 \Rightarrow \gamma_3)\right) \boxplus \widetilde{\mu}(\gamma_3 \Rightarrow \sigma[q_4, Z_2, q_6]\sigma)\right) \boxplus \widetilde{\mu}(\sigma[q_4, Z_2, q_6]\sigma \Rightarrow s)\right] \\
&\quad \bigwedge \left[\left(\bigwedge_{\gamma_3}\left(\bigwedge_{\gamma_2}\widetilde{\mu}([q_0, Z_0, q_1] \Rightarrow \gamma_2) \boxplus \widetilde{\mu}(\gamma_2 \Rightarrow \gamma_3)\right) \boxplus \widetilde{\mu}(\gamma_3 \Rightarrow \sigma\sigma[q_5, Z_1, q_1])\right) \boxplus \widetilde{\mu}(\sigma\sigma[q_5, Z_1, q_1] \Rightarrow s)\right] \\
&\quad \bigwedge \left[\left(\bigwedge_{\gamma_3}\left(\bigwedge_{\gamma_2}\widetilde{\mu}([q_0, Z_0, q_1] \Rightarrow \gamma_2) \boxplus \widetilde{\mu}(\gamma_2 \Rightarrow \gamma_3)\right) \boxplus \widetilde{\mu}(\gamma_3 \Rightarrow \sigma\sigma[q_6, Z_2, q_1])\right) \boxplus \widetilde{\mu}(\sigma\sigma[q_6, Z_2, q_1] \Rightarrow s)\right] \\
&= [a \boxplus (b \wedge c)] \wedge (a \boxplus b) \wedge (a \boxplus c) \wedge (a \boxplus b) \wedge (a \boxplus c) \\
&= a \boxplus (b \wedge c)
\end{aligned}$$

Thus $|M|_w(s) = |G^M|_w(s)$ implies $(a \boxplus b) \wedge (a \boxplus c) = a \boxplus (b \wedge c)$. □

Furthermore, let $x \odot y = (x' \boxplus y')'$, then we can dually define an automaton on lattice ordered QMV algebras by:

Definition 9.1. An $\mathcal{E}$-valued nondeterministic finite state automaton ($\mathcal{E}$NFA) is a quintuple: $M = (Q, \Sigma, I, T, \delta)$, in which
(i) Q is a finite nonempty state set.
(ii) Σ is a finite nonempty set of input symbols.
(iii) $I : Q \to \mathcal{E}$ is the initial state function.
(iv) $T : Q \to \mathcal{E}$ is the terminal state function.
(v) $\delta : Q \times \Sigma_\epsilon \times Q \to \mathcal{E}$ is the transition function, where $\delta(p, \epsilon, q) = \begin{cases} \mathbf{1}, & p = q \\ \mathbf{0}, & p \neq q, \end{cases}$ $\Sigma_\epsilon = \Sigma \cup \{\epsilon\}$.

Definition 9.2. An $\mathcal{E}$-valued deterministic finite state automaton ($\mathcal{E}$DFA) is an $\mathcal{E}$NFA whose transform function δ satisfies the condition that, for any $p \in Q$ and any $\sigma \in \Sigma$ there exists at most one $q \in Q$ with $\delta(p, \sigma, q) \neq \mathbf{0}$.

Definition 9.3. An $\mathcal{E}$-valued automaton with empty moves (an $\mathcal{E}\epsilon$NFA) is a quintuple $M = (Q, \Sigma, I, T, \delta)$ satisfying (i)–(iv) of Definition 9.1 and

(vi) $\delta : Q \times \Sigma_\epsilon \times Q \longrightarrow \mathcal{E}$ is the transition function, where $\delta(p, \epsilon, q) = \mathbf{1}$ if $p = q$.

The notions such as Σ^*, $\mathcal{H}_\Sigma$ and n-path are defined in a similar way to Section 2. Besides that, the n-path π is assigned with the function $||\pi|| : \mathcal{H}_\Sigma \longrightarrow \mathcal{E}$, such that

$$||\pi||(\sigma_1 \cdots \sigma_n) = \odot_{i=0,1,\ldots,n-1}\delta(p_i, \sigma_{i+1}, p_{i+1})$$

Definition 9.4. The $\mathcal{E}$-valued language accepted by an $\mathcal{E}$NFA M in a depth-first mode, denoted by $|M|_d$, is defined as

$$|M|_d(s) = \vee_{p,q\in Q} \vee_{\pi\in P^n_M(p,q)} I(p) \odot ||\pi||(s) \odot T(q)$$
$$= \vee_{p_i\in Q} I(p_0) \odot \delta(p_0, \sigma_1, p_1) \odot \cdots \odot \delta(p_{n-1}, \sigma_n, p_n) \odot T(p_n)$$

where $s = \sigma_1 \cdots \sigma_n \in \Sigma^*$.

Definition 9.5. The $\mathcal{E}$-valued language accepted by an $\mathcal{E}\epsilon$NFA M in a depth-first mode denoted by $|M|_d$, is defined as

$$|M|_d(s) = \vee_{||s'||=s} \vee_{p,q\in Q} \vee_{\pi\in P^n_M(p,q)} I(p) \odot ||\pi||(s') \odot T(q)$$
$$= \vee_{||s'||=s} \vee_{p_i\in Q} I(p_0) \odot \delta(p_0, \sigma'_1, p_1) \odot \cdots \odot \delta(p_{m-1}, \sigma'_m, p_m) \odot T(p_m)$$

where $s = \sigma_1 \cdots \sigma_n \in \Sigma^*$ and $s' = \sigma'_1 \cdots \sigma'_m \in \mathcal{H}_\Sigma$.

Similarly, we define the recognized language for an $\mathcal{E}\epsilon$NFA in width-first mode. Let $M = (Q, \Sigma, I, T, \delta)$ be an $\mathcal{E}\epsilon$NFA, define $\delta_w : \mathcal{E}^Q \times \Sigma_\epsilon \longrightarrow \mathcal{E}^Q$ as

$$\delta_w(X, \sigma)(q) = \vee_{p\in Q} X(p) \odot \delta(p, \sigma, q)$$

for any $X \in \mathcal{E}^Q$. Define $\delta_w(X, t\sigma) = \delta_w(\delta_w(X, t), \sigma)$ for any $t \in \mathcal{H}_\Sigma$, $\sigma \in \Sigma_\epsilon$.

Definition 9.6. The $\mathcal{E}$-valued language accepted by an $\mathcal{E}\epsilon$NFA $M = (Q, \Sigma, I, T, \delta)$ in a width-first mode, denoted by $|M|_w$, is defined as

$$|M|_w(s) = \vee_{||t||=s, t\in\mathcal{H}_\Sigma} \left(\vee_{q\in Q} \delta_w(I, t)(q) \odot T(q)\right)$$

for any $s \in \Sigma^*$.

Similarly, we can define the pushdown automata and its language. Furthermore, in a similar way, can discuss the corresponding properties of languages and automata.

Remark 9.1. Let $\mathcal{E}$ denote an orthomodular lattice. Then $\odot$ becomes the $\wedge$ operation in orthomodular lattice. It is easy to see that the language recognizability in Definitions 9.5 and 9.6 will degenerate into the orthomodular-valued recognizability in the depth-first (resp. width-first) mode by Ying in [40,41] and Lu and Zheng in [26]. Correspondingly, our main result such as Theorem 6.2 can degenerate into Theorem 7.1 (3) of [40] (page 59), Theorem 9.1 can degenerate into Theorem 72 in [41] (page 729), Theorem 9.2 can degenerate into Theorem 73 in [41] (page 730) and Theorems 9.3 and 9.5 can degenerate into corollary 75 in [41] (page 731) without using any other techniques.

10. Conclusion

A systematic theory of quantum computation based on sharp quantum logic has been recently developed by Ying [38–41] and others [26,31–34]. However, the theory cannot meet the realistic needs of open quantum systems. In order to develop a quantum computation theory characterizing open quantum system, we have developed an automata theory based on unsharp quantum logic.

In this paper, we considered two algebraic models of unsharp quantum logic. One is the lattice ordered QMV algebras, and the other is extended lattice ordered effect algebras. They are the main algebraic models for unsharp quantum logic and we call them $\mathcal{E}$-valued lattice. In this theory, unsharp quantum logic is treated as an $\mathcal{E}$-valued logic. The notions of $\mathcal{E}$-valued finite state automata and $\mathcal{E}$-valued pushdown automata, and their various variants are introduced. The classes of languages accepted by them are defined. Various properties of automata are re-examined in the framework of unsharp quantum logic, including the closure properties of regular languages and context-free languages under various operations. The Kleene theorem and equivalence between pushdown automata and context-free grammars are studied.

Combining our results with results obtained in [40,41], we have got a clear picture on the essential difference between automata theory based on quantum logic and classical automata theory. Namely, the universal validity of many fundamental properties of automata depends not only heavily on the distributive law but also on the non-contradiction law in quantum case. We can conclude that from classical automata theory to automata theory based on sharp quantum logic, and further to automata theory based on unsharp quantum logic, the power of computing theory becomes weaker and weaker. Many advantageous properties based on classical boolean logic no longer exist in automata theory based on sharp quantum logic. At the same time, many advantageous properties based on sharp quantum logic no longer exist in automata theory based on unsharp quantum logic. In order to set up an effective computing theory based on unsharp quantum logic, we need to further study unsharp quantum logic in depth with respect to its model theory and proof theory.

References

[1] G. Birkhoff, J. von Neumann, The logic of quantum mechanics, Annals of Mathematics 379 (1936) 823–843.
[2] H.P. Breuer, F. Petruccione, The Theory of Open Quantum Systems, Oxford University Press, 2002.
[3] P. Busch, H.J. Schmidt, Coexistence of qubit effects, arxiv:6802.4167v3.
[4] P. Busch, Unsharp reality and joint measurements for spin observables 33 (1986) 2253–2262.
[5] C.C. Chang, Algebraic analysis of many valued logics, Transactions of the American Mathematical Society 88 (2) (1958) 467–490.
[6] M.D. Chiara, R. Giuntini, R. Greechie, Reasoning in Quantum Theory—Sharp and Unsharp Quantum Logic, Kluwer Academic Publishers, 2004.
[7] E.B. Davies, Quantum Theory of Open System, Academic, New York, 1976.
[8] A. Dvurečenskij, S. Pulmannová, New Trends in Quantum Structures, Kluwer, Dordrecht, 2000, Ister Science, Bratislava.
[9] D.J. Foulis, M.K. Bennett, Effect algebras and Unsharp Quantum Logics, Foundations of Physics 24 (1994) 1331–1352.
[10] B. Gerla, Automata over MV algebras, in: Proc. 34th international symposium on multiple-valued logic, 2004, pp. 49–54.
[11] R. Giuntini, H. Greuling, Torward a Formal Language for Unsharp Properties, Foundations of Physics 19 (1989) 931–945.
[12] R. Giuntini, Quantum MV algebras, Studia Logica 56 (1996) 393–417.
[13] R. Giuntini, Quantum MV-algebras and commutativity, International Journal of Theoretical Physics 37 (2004) 65–74.
[14] S. Gudder, Total extensions of effect algebras, Foundations of Physics Letters 8 (1995) 243–252.
[15] S. Gudder, Quantum computers, International Journal of Theoeretical Physics 39 (2000) 2151–2177.
[16] K.-E. Hellwig, Coexistent effects in quantum mechanics, International Journal of Theoeretical Physics 2 (1969) 147–155.
[17] J.E. Hopcroft, J.D. Ullman, Introduction to Automata Theory, Languages, and Computation, Addision-Wesley, Reading, MA, 1979.
[18] K. Husimi, Studies on the foundation of quantum mechanics, Proc. Phys. Math. Soc. Japan 19 (1937) 766–789.
[19] G. Kalmbach, Orthomodular Lattices, Academic Press, London, 1983.
[20] I. Kamleitner, Open quantum system dynamics from a measurement perspective: applications to coherent particle transport and to quantum brownian motion, arXiv:1009.4349v1.
[21] I. Kaplansky, Any orthocomplemented complete modular lattice is a continuous geometry, Annals of Mathematics 61 (1955) 524–541.
[22] A. Kondacs, J. Watrous, On the power of quantum finite state automata, in: Proc. 38th Annu. Symp. on Foundations of Computer Science, Miami, Florida, 1997, pp. 66–75.
[23] F. Kôpka, F. Chovanec, D-posets, Mathematical Slovaca 44 (1994) 21–34.
[24] P. Lahti, Coexistence and joint measurability in quantum mechanics, International Journal of Theoretical Physics 42 (2003) 893–906.
[25] Y.M. Li, Finite state automata based on quantum logic and monadic second-order quantum logic, Science China Information Sciences 53 (2010) 101–114.
[26] R.Q. Lu, H. Zheng, Lattices of quantum automata, International Journal of Theoretical Physics 42 (2003) 1435–1459.
[27] G. Ludwig, Foundations of Quantum Mechanics, Vol. 1, Springer, Berlin, 1983.
[28] G.W. Mackey, Mathematical Foundations of Quantum Mechanics, Benjamin, New York, 1963.
[29] C. Moore, J. Crutchfield, Quantum automata and quantum grammars, Theoretical Computer Science 237 (2000) 275–306.
[30] A. Paz, Introduction to Probabilistic Automata, Academic Press, 1971.
[31] D.W. Qiu, Automata and grammars theory based on quantum logic, Journal of Software (2003) 23–27.
[32] D.W. Qiu, Automata theory based on quantum logic: Some characterizations, Information and Computation 190 (2004) 179–195.
[33] D.W. Qiu, M.S. Ying, Characterization of quantum automata, Theoretical Computer Science 312 (2004) 479–489.
[34] D.W. Qiu, M.S. Ying, Automata theory based on quantum logic: reversibilities and pushdown automata, Theoretical Computer Science 386 (2007) 38–56.
[35] Z. Rieanová, Generalization of blocks for D-lattices and lattice-ordered effect algebras, International Journal of Theoeretical Physics 39 (2000) 231–237.
[36] K. Svozil, Quantum Logic, Springer-Verlag, Singapore, 1998.
[37] Y. Shang, X. Lu, R.Q. Lu, Automata theory based on unsharp quantum logic, Mathematical Structures in Computer Science 19 (2009) 737–756.
[38] M.S. Ying, Automata theory based on quantum logic (I), International Journal of Theoeretical Physics 39 (2000) 985–995.
[39] M.S. Ying, Automata theory based on quantum logic (II), International Journal of Theoeretical Physics 39 (2000) 2545–2557.
[40] M.S. Ying, A theory of computation based on quantum logic (I), Theoretical Computer Science 344 (2–3) (2005) 134–207.
[41] M.S. Ying, Quantum logic and automata theory, in: Dov Gabbay, Daniel Lehmanm, Kurt Engesser (Eds.), in: Handbook of Quantum Structures and Quantum Logic, North-Holland, Elsevier, 2007.